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M. L. Clarke

A TREATISE

ON

SHADES AND SHADOWS,

AND

LINEAR PERSPECTIVE.

BY CHARLES DAVIES,

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PREFACE.

Within the last few years a great change has taken place in public sentiment, as to the importance which should be given to mathematical studies in comparison with other branches of education. Until recently it was thought that mere practical rules, unaccompanied by demonstration, were abundantly sufficient for all useful applications of mathematical science, and that the mind of the scholar could find richer nutriment in Virgil and Homer, than in the propositions of Euclid, or the sublime theories of Newton.

1-20-45 MSL

But it is auspicious to the cause of sound learning, that these opinions have given place to more rational views of education; that we are at length convinced it is better to reason than merely to remember; and that the value of an education is to be estimated by the ability which it gives to the mind of thinking profoundly and reasoning correctly.

In presenting to the public the following Treatise on Shadows and Perspective, the author cannot but flatter himself that he shall add something to the common stock of useful knowledge. The subjects treated of are certainly useful: to the architect and draftsman a knowledge of them is indispensable. To find with mathematical accuracy the lines of shade and shadow on a complicated building,—which parts are to be darkened, and which parts are to be made light in a drawing of it, is certainly a difficult problem unless it be solved on scientific principles.

The art of Perspective teaches us how to represent on a surface one or more solid bodies, in such a manner that the picture shall exhibit the same appearance as is presented by the objects themselves. It is by this art that the painter is enabled to present to the eye the almost living landscape, with its hills, its valleys, its waterfalls; and its rich foliage, varied by the beautiful tints of colour, and relieved by alternate light and shade. It is this art which has stamped the canvass with the intelligence of the human countenance, and caused it to be looked upon as the remembrancer of departed worth and the record of former times. It is this art which presents in a panorama a city in all its proportions, and causes the spectator to feel that he almost participates in its bustle and business. This art also enlarges the pleasures of sight, the sense through which the mind receives the most numerous and pleasing impressions.

Without perfect accuracy in the perspective, the proportions of objects cannot be preserved; and the skill of the draftsman, or the genius of the painter, is exerted in vain, if nature be not correctly copied.

The manner of finding the shadows of objects, and the common methods of perspective, depend on mathematical principles, and are susceptible of demonstration. The want of a work demonstrating those principles rigorously, has long been felt, and especially in the Military Academy, where the subjects have been taught by lecture for several years. If the one now submitted be found not to merit the approbation of the public, the author hopes it will at least be received with indulgence.

The author would beg leave to express his acknowledgments to two or three friends, to whom he is indebted for drafts of the diagrams; and also, to repeat his expressions of thankfulness to the Cadets for the interest they have taken in the work. But for their liberality it could not have appeared.

Military Academy, }
West Point, March, 1832. }

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SHADES AND SHADOWS.

CHAPTER I.

DEFINITIONS AND FIRST PRINCIPLES.

1. LIGHT is that, which proceeding from an object and falling upon the eye, produces the sensation of sight.

2. Bodies which emit or give out light, such as the sun, a candle, &c., are called luminous bodies.

3. Light is supposed to emanate from every point of a luminous body, and to proceed in right lines when not deflected or turned from its course.

4. The right lines, along which the light is supposed to move, are called *rays of light.*

5. Since light flows from each point of a luminous body in every direction, the rays of light drawn through a single point form a system of diverging lines. These rays are more divergent when the luminous object is near; the divergency diminishes when the distance to the luminous object is increased, and becomes nothing, or the rays become parallel, when that distance is infinite.

6. The sun, the chief source of light, is so far distant from the earth, that the rays of light coming from it may be regarded as parallel. As it is only proposed, in this elementary treatise, to find the shades and shadows of objects as they appear in nature, the rays of light, in the constructions here given, will be considered as parallel with each other.

Having considered the subject under this point of view, it will be easy to extend the principles developed, to embrace the cases in which the rays of light are convergent or divergent.

7. In the graphic constructions which follow, it becomes necessary to represent rays of light by right lines. When a line is assumed and called a ray of light, it is to be understood as merely indicating the direction of the light as it falls upon the body whose shade or shadow is considered.

8. Bodies or objects may be divided into three classes:

1°. Luminous bodies, or those which give out light; such as the sun, a candle, &c.

2°. Opaque bodies, or those which intercept the light; such as wood, stone, iron, &c.

3°. Transparent bodies, or those through which the light passes freely; such as water, glass, &c.

9. Light emanating from a luminous object, and falling upon an opaque body, is received by that part of the body which is towards the source of light. This portion of the surface, on which the light falls, is called the *illuminated part of the body*. That portion on which the light does not fall, is called the *shade of the body;* and the line which separates the illuminated part from the shade is called the *line of shade.*

Let C (Pl. 1, Fig. 1), a point in the plane of the paper, be the centre of a sphere, CB its radius, and AF, also in the plane of the paper, a ray of light. The rays of light being parallel, and falling upon the sphere in the direction AF, may be regarded as forming a cylinder, of which AC is the axis. The surface of this cylinder is tangent to the sphere in the great circle of which the line BE, drawn at right angles to AC, is the projection. —(Davies' Des. Geom. 108.)

All the rays of light which lie within the surface of this cylinder, pierce the sphere in the illuminated part. But the sphere being supposed opaque, the light is intercepted, and does not fall on the opposite hemisphere. Hence,

1°. The surface of the hemisphere BFE, towards the source of light, is the illuminated part of the sphere.

2°. The surface of the hemisphere BPE, opposite the source of light, is the shade of the sphere.

3°. The great circle of which BE is the projection, is the line of shade.

10. The shadow of an object is that part of space from which the object excludes the light of a luminous body. Hence, the opaque bodies only cast shadows.

Referring again to the sphere of which C is the centre, and to the cylinder of which AC is the axis, it follows, from the above definition, that all the space contained within the surface of the cylinder, estimated from BE, in the direction CP, is but the indefinite shadow cast by the sphere.

11. Let us now suppose a body to be placed at any distance from the sphere, and opposite to the source of light. It is plain that the sphere will prevent the light from falling upon a part of the surface of the body so placed. The part of the surface from which the light is excluded is called the *shadow on the body*; and the boundary of this shadow, the *line of shadow*.

This shadow on the body is contained within the surface of the cylinder of rays that is tangent to the sphere, and the line of shadow is the line of intersection of the surface of this cylinder with the body on which the shadow falls.

12. If a single point, in space, be supposed to intercept the light, the direction of its shadow will be the

same as that of the ray passing through it; and the shadow will lie on the opposite side of the point from the source of light.

13. If the light be intercepted by a right line, the shadow will be determined by drawing rays of light through all its points. These rays being parallel, and intersecting the same right line, are all contained in a plane passing through the line so intersected. This plane is called *a plane of rays.*

14. The shadow of a curved line is determined by drawing rays of light through all its points. These rays, being a system of parallel lines, may be regarded as forming the surface of a cylinder passing through the curve. This surface is called the *surface of a cylinder of rays.*

15. Applying what has been shown of the sphere to any opaque body, and recapitulating, in part, what has already been said, the following definitions and principles may be laid down.

1°. The illuminated part of a body is that portion of its surface on which the light falls.

2°. The shade of a body is that part of the surface from which the light is excluded by the body itself.

3°. The line of shade is the line separating the shade from the illuminated part of a body, and is the line of contact of a cylinder of rays tangent to the body.

4°. The indefinite shadow of a body is that part of space from which the body excludes the light, and is limited by the surface of the tangent cylinder of rays.

5°. The shadow *on* a body is that portion of its surface from which the light is excluded by an opaque body, between it and the source of light. The boundary of this shadow is the line of shadow.

6°. The line of shadow is the intersection of the surface of a cylinder of rays tangent to the opaque body, with the surface on which the shadow falls.

7°. The shadow cast by a point upon any surface is determined by finding where a ray of light drawn through the point pierces the surface.

8°. The shadow cast by a right line upon any surface is the intersection of a plane of rays passing through the line with the surface.

9°. The shadow of a curve upon any surface is the intersection of the surface of a cylinder of rays passing through the curve, with the surface on which the shadow falls.

10°. The line of shade upon a body is always a line of contact, and the line of shadow a line of intersection.

11°. Since the line of shadow is determined by the surface of the tangent cylinder of rays, it follows, that the line of shade is the line which casts the line of shadow on the body where the shadow falls.

12°. When the curve casting the shadow is a plane curve, and has such a position that its plane is a plane of rays, the rays of light drawn through all its points will form a plane and not a cylindrical surface. In such case, the shadow of the curve upon a surface is the intersection of its plane with the surface.

16. If a right line is tangent to a curve in space, the shadow of the right line will be tangent to the shadow of the curve. For, the plane of rays which determines the shadow of the right line will be tangent to the cylinder of rays which determines the shadow of the curve; therefore, their intersections by the surface on which the shadows fall are tangent to each other.

17. If two curves are tangent to each other in space, their shadows on any surface will also be tangent.

For, the cylindrical surfaces which determine the shadows of the curves are tangent to each other; hence, the curves in which they intersect the surface on which the shadows fall, are also tangent to each other.

18. The shade of an object being considerably darker than the illuminated part, is easily distinguished from it; and the line of shade is, in general, distinctly marked. In the drawings, the shade will be distinguished by small parallel lines, as in the hemisphere EPB (Pl. 1, Fig. 1).

Shadows also, appearing darker than that part of the same surface on which the shadow does not fall, will, in the drawings, be darkened by small parallel lines, in the same manner as the shade.

19. In the drawings to be made, two planes of projection will be used, as in Descriptive Geometry.

20. Since the projections of an object should represent the appearance which the object itself presents to the eye, situated at an infinite distance from, and in a perpendicular to, the plane on which the projection is made, only that part of its surface must be shaded in the drawing which is in the shade, and seen by the eye. The portion of the surface which is in the shade and not seen by the eye, must not be darkened in the drawing; if it were, the drawing would not be a true representation of the object.

In like manner, a shadow upon one of the planes of projection, or upon any other surface, may be concealed from the eye, in which case it is not to be represented in the drawing.

21. The same general rules are observed in making the projections of bodies as in Descriptive Geometry. All the bounding lines which are seen are made full; and all auxiliary lines either dotted or broken.

The rays of light will be represented by small broken lines.

22. To enable us to determine the shade and shadow of an object, there must be given,

1°. The position of the opaque object casting the shadow, which is determined when its projections are given.

2°. The surface on which the shadow falls.

3°. The direction of the light.

From these data the shade and shadow of any object can always be found.

The terms *plan* and *elevation* are generally used by Architects instead of the terms *horizontal-projection* and *vertical projection.* As the terms are synonymous, either may be used.

In architectural drawings it is customary to suppose the source of light on the left of the object, and the rays to have such a direction that their horizontal and vertical projections shall make angles of 45° with the ground line. When this is the case, it is plain that the rays will be parallel to the diagonal of a cube whose faces are parallel and perpendicular to the planes of projection; and since the diagonal of a cube makes an angle of 35° 16′ with the plane of either of its faces, it follows that any ray of light must make an angle of 35° 16′ with the planes of projection, when both its projections make angles of 45° with the ground line.

CHAPTER II.

APPLICATIONS AND CONSTRUCTIONS.

PROBLEM I.

To find the shadow cast by a right line upon the horizontal plane of projection.

23. Let IL (Pl. 1, Fig. 2) be the ground line, (BD, B′D′) be the given line; (A,A′) a ray of light, A being its horizontal, and A′ its vertical projection.

Through any point of the line as (B, B′), conceive a ray of light to be drawn. Its projections will be respectively parallel to A and A′, and the ray will pierce the horizontal plane at E, which will be one point in the shadow of the right line.

Through any other point of the right line as (D, D′) conceive a ray of light to be drawn. Its projections will also be parallel, respectively, to A and A′, and the ray will pierce the horizontal plane at F, which is a second point in the shadow of the given line.

Since the shadow of a right line upon a plane is a right line, the line EF is the shadow cast upon the horizontal plane by the line (BD, B′D′). The indefinite right line HG is the indefinite shadow on the horizontal plane, cast by an indefinite right line passing through the two points (B,B′) and (D,D′). It is plain that this shadow HG is the horizontal trace of a plane of rays passing through the line (BD, B′D′).

The right line (BD, B′D′) may be so situated that the whole, or a part of its shadow, will fall on the vertical plane. If it be required to find that shadow, we have only to construct the vertical trace of the plane of rays passing through the given line.

It is also apparent, that the point H, in which the line (BD, B′D′) produced, pierces the horizontal plane, is in the right line joining the points E and F, since the three points are all in the trace of the same plane.

Hence we conclude, that *the point in which a right line pierces a plane is one point of the indefinite shadow of the line upon the plane.* And, extending the principle, *the indefinite shadow cast by a right line upon any surface, passes through the point in which the right line, produced if necessary, pierces the surface.*

24. When a right line is parallel to the plane on which the shadow falls, the shadow will be parallel to the line itself. For if the line and its shadow be not parallel, they would, if produced, intersect, which they cannot do, since a line cannot intersect a plane to which it is parallel.

Construction of the figure. Draw on the paper the indefinite line IL, to represent the intersection of the planes of projection, or the ground line. Then draw the lines BD, B′D′ to represent the projections of the given line; also, A and A′ to represent the projections of the ray of light. This being done, through B and D, draw lines parallel to A. Through B′ and D′ draw lines parallel to A′. At the points E′ and F′, erect in the horizontal plane perpendiculars to the ground line, and mark the points E and F in which they meet the parallels first drawn. Draw the line EF, which is the shadow required.

PROBLEM II.

Having a rectangular pillar with a rectangular abacus placed upon its upper face; it is required to find the shadow cast by the abacus on the faces of the pillar, and the shadow cast by the abacus and pillar on the horizontal plane.

25. Let the square *ab* (Pl. I. Fig. 3) be the horizontal projection of the pillar, *cd* its vertical projection, EC the horizontal projection of the abacus, and E′G its vertical projection. The direction of the light is shown by the ray (A, A′).

Let us first find the shadow of the line (BC, FG), on the front face of the pillar. Through (B, F) a point of the line, conceive a ray of light to be drawn; the two projections are respectively parallel to A and A′. The point (*h*,*h*′), in which this ray pierces the front face of the pillar, is one point in the shadow of the line (BC, FG) on that face. But the line (BC, FG) being parallel to the front face, the shadow is parallel to the line itself; hence *h*′*m*, drawn parallel to the ground line, is the vertical projection of the required shadow. The horizontal projection is in the line *hb*.

The line of which BE is the horizontal projection, and which is vertically projected at F, casts a shadow upon the front face of the pillar, on the face *na*, and on the horizontal plane.

Conceive a plane of rays to be passed through this line—this plane is perpendicular to the vertical plane of projection, and its vertical trace is F*n*′*h*′, parallel to the vertical projection of the light; and *n*′*h*′ is also the projection of its intersection with the front face of the pillar: therefore *n*′*h*′ is the vertical projection of the shadow cast on the front face of the pillar by a part of the line (BE, F)—the horizontal projection of the same shadow is *nh*. Drawing through *n* the line *pm* parallel to the horizontal projection of the light, gives B*p* the horizontal projection of that part of the line which casts a shadow on the front face.

The part of the line from *p* to *q* casts a right line of shadow on the face *na*, which shadow is vertically projected at *n′*. The part *q*E casts a shadow on the horizontal plane. In finding the shadow of the abacus on the horizontal plane, we will begin with the point (C, D′).

Through this point suppose a ray of light to be drawn. Such ray will pierce the horizontal plane at H. Through (D, D′) conceive a ray also to be drawn. Such ray will pierce the horizontal plane at I. Hence HI is the shadow cast upon the horizontal plane by the line (CD, D′).

But I is also the shadow on the horizontal plane of one point of the line (ED, E′D′); and since this line is parallel to the plane, IL drawn parallel to ED, is its indefinite shadow. The shadow is limited by the line EL, drawn through E, parallel to the horizontal projection of the light.

Through the point (C, G) conceive a ray of light to be drawn. It pierces the horizontal plane at *i*. The line H*i* is the shadow cast by the vertical line (C, GD′) of the abacus.

Through the point *i* draw *i*K parallel to BC, and produce it till it meets *k*K, drawn through *b*, parallel to the horizontal projection of the light. Then K*i* is the shadow cast by (*k*C, *k′*G) a part of the front and lower line of the abacus. The ray of light through the point (*k*, *k′*) touches the edge of the pillar at *m*, and pierces the horizontal plane at K.

From K to *b* the shadow is cast by the part of the edge (*b*, *md*).

Returning to the point L, L*f* is the shadow cast by the line (E, E′F), *fg* the shadow cast by the line (E*q*, F), and *ga* the shadow cast by that part of the edge between *n*′ and the horizontal plane.

The light falls on three faces of the abacus, viz: the upper face, whose vertical projection is the line E′D′; the side face whose vertical projection is the line E′F; and the front face whose vertical projection is the rectangle FD′. The three remaining faces are in the shade, but neither of them is seen in either projection.

The part of the front face of the pillar on which the shadow of the abacus falls, and the shadow of the pillar and abacus on the horizontal plane, are shaded. The part of the shadow on the horizontal plane which falls under the abacus cannot be seen in the horizontal projection, and is therefore not shaded.

It may not be out of place to remark, that the lines of shadow which have been determined, are but the intersections of planes of rays passing through the lines casting the shadows, with those planes on which the shadows fall.

Thus *h*′*m* is the trace, on the front face of the pillar, of a plane of rays passing through (BC, FG). The shadow *n*′*h*′ is the trace, on the front face of the pillar, of a plane of rays passing through the line (EB, F); HI is the trace, on the horizontal plane, of a plane of rays passing through (CD, D′); H*i* is in the horizontal trace of a vertical plane of rays passing through the vertical line (C, D′G). The shadow *i*K is in the horizontal trace of a plane of rays passing through (BC, FG). The shadow K*b* is in the horizontal trace of the vertical plane of rays passing through the edge (*b*, *dm*); and this plane cuts off the part ($kC, k'G$) of the line ($BC, F'G$), which casts the shadow Ki on the horizontal plane. Similar explanations may be given of the other lines of shadow on the horizontal plane.

PROBLEM III.

It is required to find the shadows cast by the cornices of the chimneys of a house, on the faces of the chimneys; the shadows cast by the cornices and chimneys on the roof of the house, and the shadows cast by the roof on the walls, and on a horizontal plane at a given distance below the eaves.

26. Let the rectangle BCEA (Pl. 2) be the horizontal projection of the outer lines of the eave-trough. These lines are contained in a horizontal plane, and are vertically projected in the line A'C'. The lines of the inner rectangle are the intersections of the outer faces of the walls with the horizontal plane; they are made broken in projection, being concealed by the roof. The lines of the middle rectangle are the horizontal projections of the eaves of the roof. The eaves are supposed to be in the same horizontal plane with the upper line of the eave-trough, and are therefore vertically projected in the line A'C'.

The line in which the side roofs intersect each other, and the lines in which the side roofs intersect the end roofs, are represented by full lines in horizontal projection.

With regard to the chimneys, the lines of the outer rectangle, as *ifhq*, are the horizontal projections of the outer lines of the cornice; the lines of the middle rectangle are the intersections of the outer faces of the chimneys with the roofs; and the inner rectangle is the projection of the flue or hollow part of the chimney.

The vertical projections of the roofs and chimneys can be understood from the figure, excepting, perhaps, the manner of determining the lines in which the front and back faces of the chimney in the front roof, intersect the roof. After the horizontal projection of the chimney is made, these lines are thus found:

The faces of the chimney intersect the front roof in lines parallel to the ground line. Producing the horizontal projection of the front face till it meets in G, the line of the end roof, we obtain IG, the horizontal projection of the line of intersection. But G is vertically projected in G′, therefore GT′ drawn through G′ parallel to the ground line, is the vertical projection of the line in which the front face of the chimney intersects the roof. For similar reasons, the line drawn through H′ parallel to the ground line, is the vertical projection of the line in which the back face of the chimney intersects the roof.

In finding the shadows, let us begin with the chimney of the end roof.

Through (f, f''), a point of the lower line of the cornice, conceive a ray of light to be drawn. It pierces the front face of the chimney in the point (g, g'); and the line $g'p$ is the vertical projection of the shadow cast by $(fh, f''h'')$ on the front face of the chimney. The line og' is the vertical projection of the shadow cast on the front face by a part of the line $(fi. f'')$. A similar construction gives the shadow cast on the front face of the other chimney.

To find the shadows which the chimneys cast on the roofs.

Through the line (hq, h') conceive a plane of rays to be passed. Since the line (hq, h') is perpendicular to

Plate I.

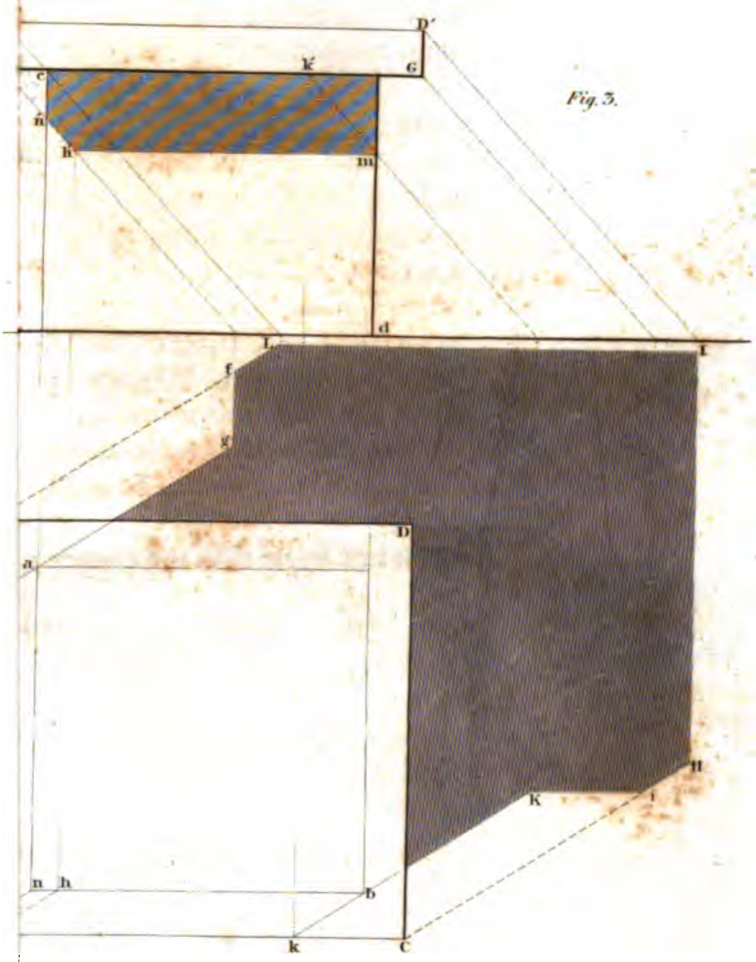

Fig. 3.

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the vertical plane of projection, the plane of rays through it will also be perpendicular to the vertical plane; and hence its vertical trace will be the line $h'k'F''$ parallel to the vertical projection of the light. This plane of rays intersects the horizontal plane of the eaves in a line which is perpendicular to the vertical plane at F'', and of which $\mathrm{F}'\mathrm{F}$, perpendicular to the ground line, is the horizontal projection. It also cuts the line of intersection of the side roofs in a point whose vertical projection is k' and whose horizontal projection is k; and hence kF, kF' are the horizontal projections of the intersections of the plane of rays through (hq, h') with the side roofs; which intersections are the indefinite shadows cast by (hq, h') on these roofs. The shadows are limited in horizontal projection by the lines hn and qm drawn parallel to the horizontal projection of the light. Therefore kn and km are the horizontal projections of the shadows cast on the side roofs by the line (hq, h').

It may be here remarked, that *all planes of rays which are perpendicular to the vertical plane of projection are parallel to each other, and will consequently intersect the side roofs in lines parallel to* $k\mathrm{F}'$ *and* $k\mathrm{F}$.

Drawing through m the line ms, parallel to the ground line, and through i, is parallel to the horizontal projection of the light, gives ms for the shadow of the line $(iq, f'h')$ on the side roof. The line us is the horizontal projection of the shadow cast by the perpendicular $(i, f'f'')$. This shadow is in the trace of a vertical plane of rays through $(i, f'f'')$, and is limited at u by the intersection of a plane of rays through (fi, f'') with the side roof.

The line uv, lying in this intersection, is the shadow cast by a part of the line (fi, f'') on the side roof. The shadow at v is cast by that point of the back edge of the chimney which is vertically projected at o.

From v the shadow is cast by the back edge of the chimney.

Returning to the shadow cast by the point (h, h'), we find, first, the shadow nx, which is cast by the perpendicular $(h, h'h'')$, and then draw through x a parallel to the ground line; this parallel is the indefinite shadow cast by the line $(fh, f''h'')$ on the side roof. We next find the point in which a ray of light through the point whose vertical projection is p, pierces the end roof, which is at (t, t'). Joining the point t with the point in which the parallel through x meets the intersection of the side and end roofs, gives the shadow cast by the line $(fh, f''h'')$ on the end roof. From t the shadow is cast by the edge of the chimney.

We find the vertical projection of this shadow by projecting its points into the vertical traces of the planes to which they respectively belong. Thus m is vertically projected at m', s at s', u at u', v at v', y at y', k at k', n at n', and x at x'. That part of the shadow which is on the front roof is all that can be seen in vertical projection. The boundary of that on the back roof is therefore dotted.

The shadows of the other chimney are found in a manner so entirely similar as not to require a particular explanation.

Let us now find the shadows on the walls of the house.

Through (BC, ac), the upper line of the convex part of the eave-trough, conceive a plane of rays to be passed. Its trace on the front wall limits the shadow cast by the eave-trough. The point (b, b') is in the trace of this plane, and $b'e$ is the required line of shadow. The shadow $b'z$ is cast by the corresponding line of the end eave-trough, which also casts a shadow on the end wall that is vertically projected at z.

To find the shadow cast by the house on the horizontal plane of projection.

Through (CE, C′), the upper line of the end eave-trough, conceive a plane of rays to be passed. The line LE′ is its trace on the horizontal plane, and consequently the shadow of the line (CE, C′).

The small vertical line (C, *c*C′) casts its shadow L*c*′ in the line CL. Through *c*′ draw *c*′*d* parallel to the ground line, and produce it till it meets D*d* drawn parallel to the horizontal projection of the light. Then *dc*′ is the shadow cast by the line (DC, D′*c*), and the ray through (D, D′) touches the corner of the house at *e*, and pierces the horizontal plane at *d*.

The line *a*′E′, drawn through E′ parallel to the ground line, and limited by A*a*′ drawn parallel to the horizontal projection of the light, is the shadow cast on the horizontal plane by the line (AE, A′C′). The line (A, *a*A′) casts the shadow *a*′*r*; a part of the line (BA, *a*) casts the shadow *r*′*r*′: from *r*′ the line of shadow is cast by the corner of the house below *z*.

The shadow on the horizontal plane, which lies under the eave-trough and cornice, is not seen in horizontal projection.

PROBLEM IV.

Having given an oblique cylinder, and the direction of the light, it is required to find the shade on the exterior surface, the shadow of the upper circle on the interior surface, and the shadow of the cylinder on the horizontal plane.

27. Let (AB, A′B′), (Pl. 3. Fig. 1), be the axis of the cylinder, and suppose the projections to be made as seen in the figure.

If we suppose two tangent planes of rays to be drawn to the surface of the cylinder, the elements of contact will separate the dark from the illuminated part of the surface, and will consequently be lines of shade.

In order to pass these planes, we first pass a plane of rays through the axis of the cylinder, to which the tangent planes will both be parallel.

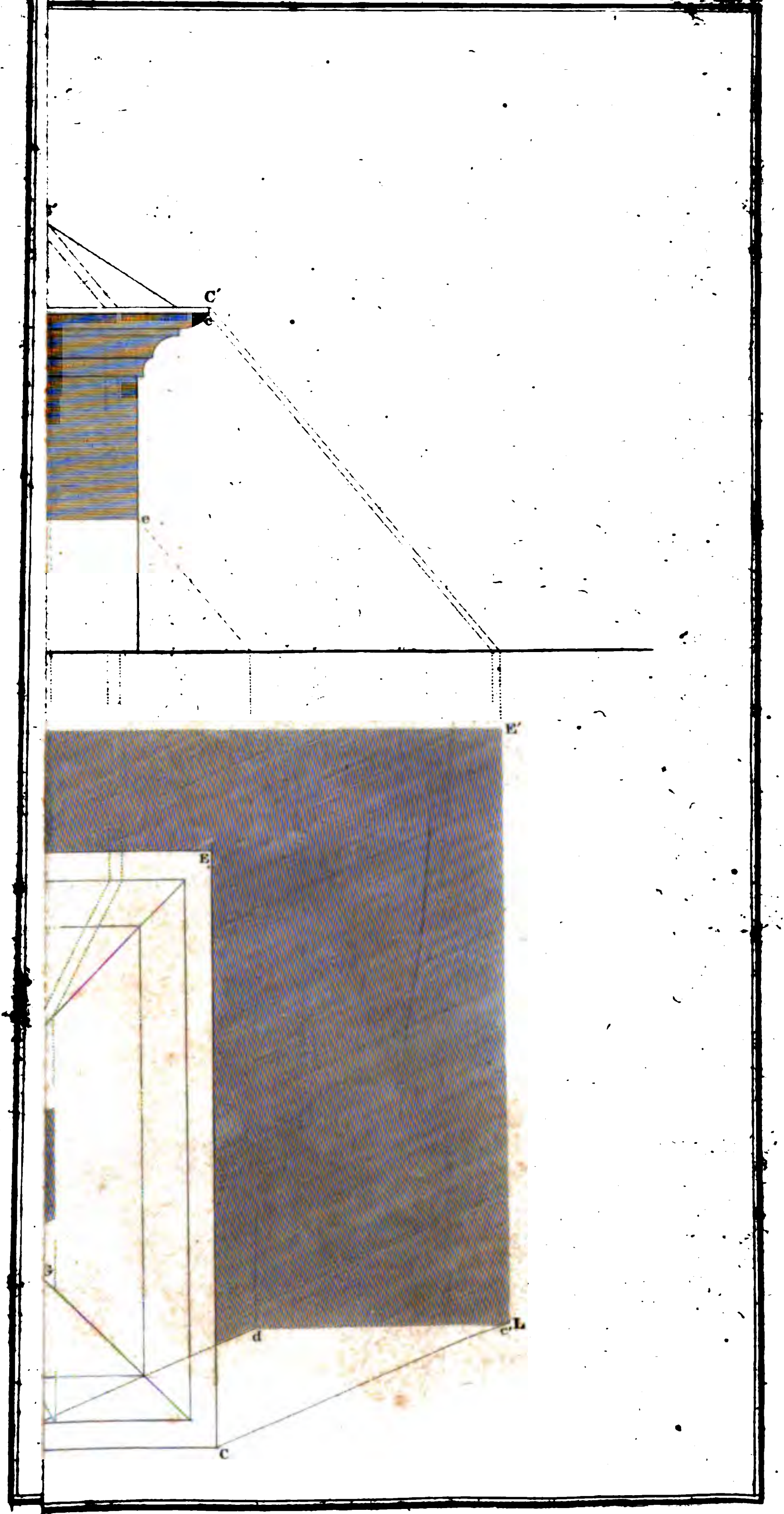
Through (B,B′), a point of the axis, conceive a ray of light to be drawn—it pierces the horizontal plane at C. The axis of the cylinder pierces the horizontal plane at A—therefore AC is the horizontal trace of a plane of rays passing through the axis of the cylinder.

Let two lines be now drawn tangent to the base of the cylinder, and parallel to AC. These lines are D*e* and G*f*″, and are also the horizontal traces of the planes of rays tangent to the surface of the cylinder. The elements of contact pierce the horizontal plane at D and G. Hence, DE and GF, drawn parallel to AB, are the horizontal projections of the elements of contact: and by projecting D and G into the ground line at D′ and G′, and E and F into the vertical projection of the upper circle at E′ and F′, we determine D′E′ and G′F′, the vertical projections of the elements of contact. The points D, A and G, are in the same straight line perpendicular to AC; and DHG is a semicircle.

The light falls on the half of the exterior surface corresponding to the semicircle DHG—the remaining half is in the shade. The lines (DE, D′E′,) (GF, G′F′) are lines of shade, and being also elements of the surface, are generally called elements of shade.

In the horizontal projection, we do not see the shade which is on the under surface of the cylinder—therefore, we darken only that part which is included between the elements DE and LN. In the vertical projection, we

Plate 2.



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see that part of the shade included between the element G′F″, and the element P′K′, only that portion of the surface, therefore, is shaded.

To find the shadow cast by the upper circle on the interior surface.

This shadow begins at the points (E, E′), (F,F″) in which the elements of shade meet the upper circle, and is cast by the semicircle whose horizontal projection is F*da*E.

If the surface be intersected by planes of rays parallel to the axis of the cylinder, each plane so drawn will intersect the surface in two rectilinear elements. If then, through the upper extremity of the element towards the source of light, a ray be drawn, it will be contained in the plane of rays, and will, consequently, intersect the other element lying in that plane—the point of intersection is a point of shadow on the interior surface. But the horizontal traces of all planes of rays parallel to the axis of the cylinder are parallel to AC, the trace of the plane of rays through the axis. Therefore, draw any line, as IP, parallel to AC, to represent the trace of a plane of rays parallel to the axis. This plane intersects the surface in two elements, whose horizontal projections are I*d* and PK. Through *d*, the upper extremity of the element towards the source of light, draw *df* parallel to the horizontal projection of the light —the point *f*, in which it meets PK, is the horizontal projection of a point of shadow on the inner surface. The vertical projection of this point is found by projecting P into the ground line, drawing the vertical projection of the element on which the shadow falls, and noting its intersection with the perpendicular to the ground line, drawn through the point *f*; or, by projecting the point *d* into the upper base, drawing through

d' the vertical projection of a ray of light, and noting its intersection f' with the vertical projection of the element on which the shadow falls.

If it be required to find the shadow on any particular element, as the one, for example, whose horizontal projection is LN, we have merely to pass a plane of rays through the element, and determine the other element towards the source of light, in which it intersects the surface. Then, through the upper extremity of the element casting the shadow, draw a ray of light, and where it meets the given element, is the required point of the shadow. If we take the element whose horizontal projection is LN, RL is the horizontal trace of the plane of rays passing through it—the element towards the source of light, in which this plane intersects the surface, is horizontally projected in Rb, and g is the horizontal projection of the point of shadow cast by its upper extremity.

At g, the horizontal projection of the curve of shadow is tangent to the line LN: for the vertical plane of which LN is the horizontal trace, is tangent to the vertical cylinder, which projects the curve of shadow, along the element that pierces the horizontal plane at g.

The lowest point of the curve of shadow, or that point which is farthest from the upper circle, is found by passing a plane of rays through the axis. For, since all the rays of light make equal angles with the horizontal plane, or with the plane of the upper base, the distance of the points of shadow below the upper base will increase, as the distance between the element casting and the element receiving the shadow, is increased. But, the plane of rays through the axis intersects the surface in elements farther from each other than the elements cut out by any other of the parallel planes of rays. Hence, it determines the lowest point of shadow, which is (h, h').

Having found as many points of the shadow as may be necessary, let the projections of the curve be accurately described.

In horizontal projection, the curve of shadow is seen from F until it crosses the horizontal projection of the upper circle, thence to the point E it is concealed by the surface of the cylinder. The part of the inner surface lying above the curve of shadow, receives the light; the part lying below it is in the shadow. That part of the surface which is in the shadow, and seen, is shaded in the drawing. With respect to the vertical projection, no part of the curve of shadow is seen, since it lies entirely on the inner surface of the cylinder.

To find the shadow of the cylinder on the horizontal plane.

The lines of the cylinder which cast shadows on the horizontal plane are, the two elements of shade, and the half of the upper circle which is opposite the source of light, and of which ENF is the horizontal projection.

The shadows cast by the elements of shade fall in the horizontal traces of the tangent planes of rays—hence, we may consider De and Gf'', as the indefinite shadows of the elements of shade.

If through the upper circle of the cylinder we conceive a cylinder of rays to be passed, its axis will be a ray of light passing through the centre (B, B'), and will pierce the horizontal plane at C. But, since the plane of the upper circle and the horizontal plane are parallel, they will intersect the surface of the cylinder of rays in equal circles. Therefore, a circle described with C as a centre, and a radius equal to B'K', will be the shadow cast by the upper circle on the horizontal plane. The shadows cast by the elements of shade will be tangent to this circle of shadow at the points f'' and e; which points are found by drawing a line through C perpendicular to AC, or by drawing the horizontal projections of rays through the points F and E.

Although we have spoken of the shadow cast on the horizontal plane by the whole of the upper circle, yet it is obvious that the semicircle which is towards the source of light, and which casts a shadow on the inner surface of the cylinder, cannot cast a shadow on the horizontal plane, unless we suppose the surface to be transparent, so that the light may not be intercepted by it.

The part of the shadow on the horizontal plane which is concealed by the horizontal projection of the surface, is not shaded in the drawing.

28. The shadow of a circle, or indeed of any curve, on a plane to which it is parallel, is an equal circle or curve. But the shadow of a circle on a plane to which it is not parallel, is, in general, an ellipse. The shadow cast by the centre of the circle is the centre of the ellipse, and the shadow cast by any diameter of the circle is a diameter of the ellipse. Two diameters of an ellipse are said to be conjugate when either of them is parallel to tangent lines drawn through the vertices of the other. *If any two diameters be taken in a circle at right angles to each other, their shadows will be conjugate diameters of the ellipse of shadow.*

For, if through the vertices of either diameter, tangent lines be drawn to the circle, they will be parallel to the other diameter; hence, their shadows on any plane will be parallel to the shadow of this latter diameter. But the shadows of the tangents will be tangent to the shadow of the circle (16), that is, to the ellipse of shadow. Therefore, the shadow of one diameter is parallel to the tangents drawn through the vertices of the shadow of the other. Hence the shadows cast by two diameters of a circle at right angles to each other, are conjugate diameters of the ellipse of shadow.

29. It is often required to construct ellipses when their conjugate diameters only are known. The construction is easiest made by finding the axes of the curve.

We shall, therefore, give a problem, the principles of which are found in Crozet's Conic Sections, for *finding the axes of an ellipse when two of its conjugate diameters are given.*

Let AB and DS (Pl. 3, Fig. 2), be two conjugate diameters of an ellipse. Through either vertex, as D, of either diameter, draw a parallel EF to the other diameter. At D erect in a plane perpendicular to AOD, DO′ perpendicular to EF, and make it equal to half the parallel diameter AB. With O′ as a centre, and radius O′D, describe the circle C*a*D*b*.

Now we may regard the ellipse whose conjugate diameters are AB and DS, as the shadow of the circle C*a*D*b*.

The ray of light through the centre O′ pierces the horizontal plane at O; the diameter *ab* casts the shadow AB, and the diameter CD the shadow DS.

It is now required to find two diameters of the circle C*a*D*b* at right angles to each other, whose shadows shall also be at right angles; for, the conjugate diameters of an ellipse, which are perpendicular to each other, are the axes.

To find these diameters it is necessary to construct two semicircles having a common diameter in the line EF, one lying in the horizontal plane and passing through O, the other in the vertical plane and passing through O′.

Produce O′D and make DP equal to it. Join OP and bisect it by a perpendicular line—the point N, where the perpendicular meets EF, is the common centre of the two semicircles. Let them be described with the radius NO, or NO′. Then draw the radii O′gE and O′lF. These radii are at right angles to each other; and so are their shadows EO and FO. Hence, GO and LO are the semi-axes of the ellipse. The extremities G and L are determined by drawing rays through g and l, the extremities of the perpendicular radii.

PROBLEM V.

To find the shadow of a rectangular abacus on a cylindrical column; and also, the shade of the column.

30. Let the semicircle *zgp* (Pl. 3, Fig. 3) be the plan of a semi-column, and *op′dc′* its elevation; the rectangle *cp* the plan of the abacus, and *c″d′* its elevation.

Through the lower line (*cz*,*c′*) of the abacus, conceive a plane of rays to be passed. This plane will be perpendicular to the vertical plane of projection, and its intersection with the surface of the column will be the shadow cast by the line (*cz*, *c′*), (15).

Through *c′* draw *c′i′* parallel to the vertical projection of the light. This line is the vertical trace of the plane of rays passed through the line (*cz*, *c′*), and it is also the vertical projection of the indefinite shadow cast on the column. (Des. Geom. 82.)

But the ray of light through the point (*c*, *c′*) pierces the surface of the column at (*i*, *i′*), therefore *c′i′* is the vertical projection of the shadow cast on the column by the line (*cz*, *c′*). The shadow itself is an ellipse, and is horizontally projected in the arc *zi*.

To find the shadow cast by the line (*cd*, *c′d′*):

The shadow cast by the point (*c*,*c′*) is already found at (*i*,*i′*).

Let the surface of the column be intersected by planes of rays perpendicular to the horizontal plane. These planes will cut the line (*cd*, *c′d′*) of the abacus in points, and the surface of the column in right lines—then, drawing through the points of the abacus rays of light, where they meet the elements of the column are points of the shadow.

The line *bk*, drawn parallel to the horizontal projection of the light, is the horizontal trace of a vertical plane of rays, and this plane determines (*k*, *k′*) a point of the shadow. The points (*l*, *l′*) (*g*, *g′*) (*q*, *q′*) and (*n*, *n′*) are found in a similar manner.

The vertical plane of rays, whose horizontal trace is EF, is tangent to the column. The ray, therefore, through the point (E, E′,) touches the column at (*n*, *n′*). At this point the shade begins, and the line of shade passes down the column vertically, being the element in which the plane EF is tangent to the column.

The part of the surface of the column which is above the line of shadow, and the portion which is in the shade, are shade in the drawing.

PROBLEM VI.

To find the shadow cast by a cylindrical abacus on a cylindrical column; and the shade on the abacus and column.

31. Let *nsr* (Pl. 6, Fig. 2) be the plan of a semi-column, *fr′* its elevation—*cmd* the plan of the abacus, and *c′b* its elevation.

Let the surface of the column be intersected by vertical planes of rays. These planes will intersect the lower circle of the abacus, which casts the line of shadow on the column, in points, and the surface of the column in right lines. Through the points of the abacus let rays of light be drawn—the points in which they intersect the elements of the column are points of the line of shadow. The plane of rays, whose horizontal trace is *hh*, parallel to the horizontal projection of the light, determines the point of shadow (n, n'); and the points of shadow (o, o'), (q, q') (s, s') (t, t') (u, u') and (v, v') are found in a similar manner.

The vertical plane of rays, whose horizontal trace is *mv*, is tangent to the column along the element of shade. The shade begins at the point (v, v').

The shade on the abacus is found by drawing a plane of rays which shall be tangent to it—the line *p* drawn tangent to *cmd* and parallel to the horizontal projection of the rays of light, is the horizontal trace of the tangent plane—the element of contact is the element of shade.

The part of the surface above the line of shadow, as well as that portion of it which is in the shade, is shaded in the drawing.

PROBLEM VII.

Having given a cylindrical abacus and the direction of the light, it is required to find the shadow of the abacus on a vertical plane, or wall.

32. Let *anbp* (Pl. 6 Fig. 3) be the plan, and $a'c'd'b'$ the elevation of the abacus, the abacus touching the vertical plane on which the shadow falls in the element $(p, n'o)$.

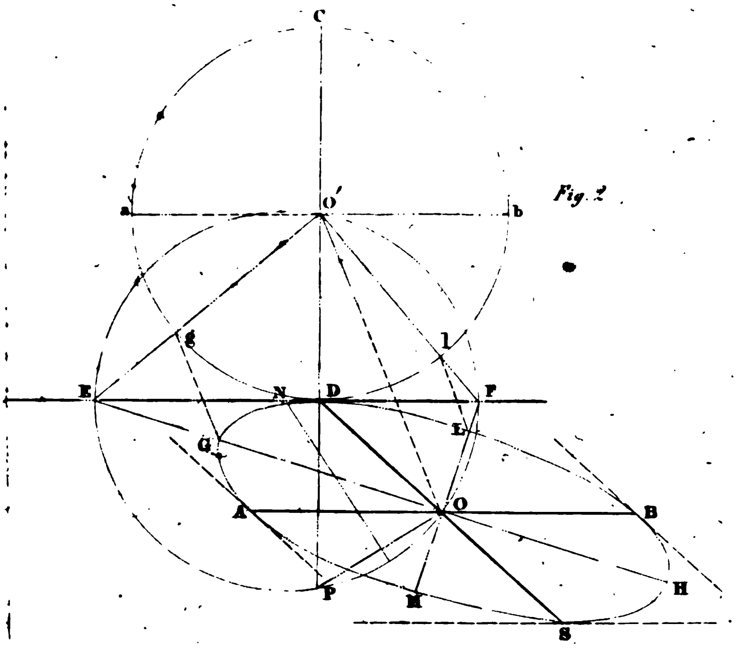

Fig. 2

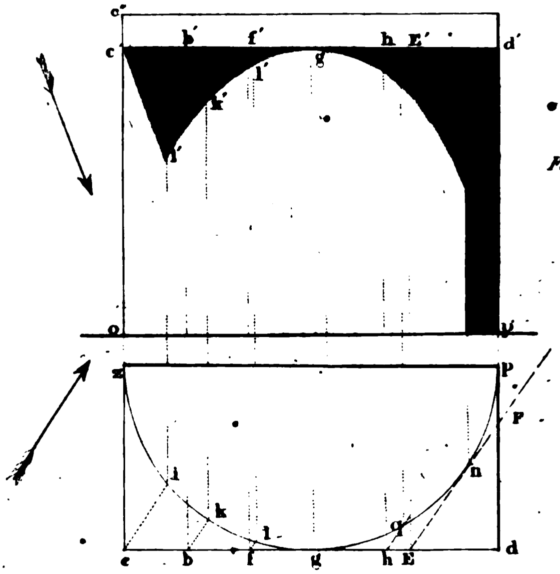

Fig. 3

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Suppose two squares to be constructed, the one circumscribing the upper circle of the abacus, the other the lower. The square $cdgl$ is the projection of both the squares on the horizontal plane.

Let two planes of rays be drawn tangent to the abacus. The lines tt'' and ii'', drawn parallel to the horizontal projection of the ray of light and tangent to the circle $nbpa$, are their traces; and $(t, t's)$, (i, yx) are the elements of contact, and consequently the elements of shade (27).

Now the lines of the abacus which cast lines of shadow on the vertical plane, are, 1st, the upper semicircle $(tbi, b't'y)$; 2dly, the two elements of shade $(t, t's)$ (i, yx); and 3dly, the lower semicircle $(iant, c'xs)$.

Through (z, n'), the centre of the upper circle of the abacus, let a ray of light be drawn,—the point z', in which it pierces the vertical plane, is the centre of the ellipse of shadow cast by the upper circle. The ray through (n, n') pierces the vertical at p, hence $n'z'p$ is the shadow cast by the diameter (np, n'). Drawing rays through (b, b') and (a, a') determines w and v, the extremities of the shadow cast by the diameter $(ab, a'b')$. Hence $n'p$ and vw are conjugate diameters of the ellipse of shadow (28).

Drawing rays of light through the points (d, b') and (c, a') determines $b'f$ the shadow cast by the tangent (dg, b'), and ef, the shadow cast by the tangent $(cd, a'b')$, and $a'e$, the shadow cast by the tangent (cl, a'). But since these lines are all tangent to the upper circle of the abacus, their shadows will be tangent to the ellipse of shadow (16); and the same may be shown of the lower circle of the abacus and its tangents.

The ellipse of shadow cast by the lower circle of the abacus is easily found. The point z'' is its centre, oq,

$x''y'$ are its conjugate diameters, and the lines $d''h$, $d''c'$, $c'g$, and gh are tangent to it. The elements of shade cast the shadows $t''u$ and $i''l$. Hence, the lines of shadow on the vertical plane, are—the semi-ellipse $lx'qu$, cast by the lower semicircle $(iant, c'xs)$; the right line ut'', cast by the element $(t, t's)$; the semi-ellipse $t''wn'i''$, cast by the upper semicircle $(tbi, b't'y)$, and the right line $i''l$ cast by the element of shade (i, yx).

PROBLEM VIII.

Having given the frustum of an inverted cone, it is required to find the shadow cast by the upper circle on the inner surface, and the shadow cast by the frustum on the plane of the lower base.

33. Let the circle CLHD (Pl. 4) be the horizontal projection of the upper circle of the frustum, and G′H′ its vertical projection. Let the circle described with the centre A and radius AK be the horizontal, and I′K′ the vertical projection of the lower base. The axis of the cone being supposed perpendicular to the horizontal plane, is projected on the horizontal plane at A, and on the vertical plane in the line A′A′ perpendicular to the ground line, and G′I′K′H′ is the vertical projection of the frustum. Producing the line G′I′ till it intersects A″A′, gives A′, the vertical projection of the vertex of the cone of which the frustum is a part.

To find the shadow on the inner surface.

If two tangent planes of rays be drawn to the cone, the points in which the elements of contact meet the upper circle are the points where the shadow on the inner surface begins.

Through (A, A′), the vertex of the cone, let a ray of light be drawn. Such ray pierces the plane of the upper base of the frustum in the point (B, B′). Through the point (B, B′), suppose two lines to be drawn tangent to the upper circle of the frustum. These tangents are the traces, on the plane of the upper base, of two planes of rays drawn tangent to the cone. The lines BC, BD, drawn tangent to the circle GDHL, are the horizontal projections of these traces; and (D, D′), (C, C′) are the points at which the shadow on the inner surface begins.

Let the surface of the cone be now intersected by planes of rays passing through the vertex. Each secant plane so drawn will intersect the surface in two elements; the element towards the source of light will cast the shadow, and the other will receive it. Each plane will also contain the ray of light passing through the vertex of the cone, and consequently its trace on the plane of the upper base will pass through the point (B, B′).

Draw any line, as B*a*L, to represent the trace of such a plane. The horizontal projections of the elements in which it intersects the surface of the cone, are AL and A*a*. Projecting L and *a* into the upper base at L′ and *a*′, and joining these points with A′, the vertical projection of the vertex of the cone, gives the vertical projections of these elements.

Through (*a*, *a*′) the upper extremity of the element towards the source of light, conceive a ray of light to be drawn; the point (*b*, *b*′) in which it meets the element (AL, A′L′) is a point of the curve of shadow.

If we suppose the surface of the frustum to be produced below the plane E′F′, the shadow that would fall on the part of the surface below this plane is easily found. The lowest point is determined by passing a plane of rays through the axis of the cone. The trace of this plane, on the upper base of the frustum, is horizontally projected in the line BA*d*. The line A′*d*′ is the vertical projection of the element which receives the shadow, and (*p*, *p*′) is the point of shadow cast by (*h*, *h*′).

By passing a plane of rays through the element (AH, A′H′) we shall find the point at which the vertical projection of the shadow is tangent to the element A′H′. The horizontal projection of the trace of this plane, on the upper base of the cone, is B*k*H. Projecting *k* into the vertical plane at *k*′, and drawing *k*′*l*′ parallel to the vertical projection of the light, gives *l*′ for the point of tangency. The horizontal projection of the point is found by projecting *l*′ into the horizontal projection of the element at *l*. Having found as many points of the curve as are necessary to describe it accurately, let its projections be made as in the figure.

It is to be observed, that the shadow on the inner surface of the frustum intersects the plane of the lower base at the points (*c*, *c*′), (*f*, *f*′); and that the curve of shadow below this plane is found on the supposition that the plane does not intercept the light, and that the frustum is produced below it.

If the plane of the lower base be supposed to intercept the light, there will be no shadow on the surface of the cone below it; for the plane itself will receive the shadow cast by the upper circle.

Through the centre (A, A″) of the upper circle, conceive a ray of light to be drawn. It pierces the plane of the lower base at (F, F′). A circle described with F as a centre, and radius equal to the radius of the upper base, is the horizontal projection of the shadow cast by the upper circle on the plane E′F′. The arc *cqf* falls within the circle of the lower base, and therefore is the line of shadow, within the surface, on the plane of that base. The circle described with the centre F will pass through *c* and *f*, the horizontal projections of the points in which the curve of shadow intersects the plane E′F′.

The shadow cast by the frustum on the plane of the lower base is limited by the traces of the two tangent planes of rays that determine the elements of shade. These planes contain the ray of light drawn through the vertex; hence their traces pass through the point (E, E′) in which this ray pierces the plane E′F′; and E*n*, E*m* drawn through E and tangent to the circle I*f*K*c* are their horizontal projections.

The shadows cast by the elements of shade on the plane E′F′ begin at the points in which they pierce it, and terminate at the points *m* and *n*, where the shadows are tangent to the shadow cast by the upper circle. The points *m* and *n* may be determined by drawing horizontal projections of rays through the points C and D, and noting their intersections with the shadow cast by the upper circle.

That part of the shadow on the plane E′F′ which is under the surface of the cone is not seen in horizontal projection.

For the purpose of showing the appearance of the shadow on the inner surface, we will suppose that part of the frustum which is in front of the vertical plane GAH to be removed, and then represent in vertical projection the remaining semi-frustum, as it appears to the eye.

That part of the surface which lies between the element (AG, A′G′) and the curve of shadow, being in the shadow, is darkened in vertical projection.

PROBLEM IX.

Having given a sphere in space, and the direction of the light, it is required to find the curve of shade, and the shadows cast on the planes of projection.

34. Let the centre of the sphere be taken at equal distances from the planes of projection; let A (Pl. 5) be its horizontal and A′ its vertical projection; and suppose the projections of the light to make equal angles with the ground line.

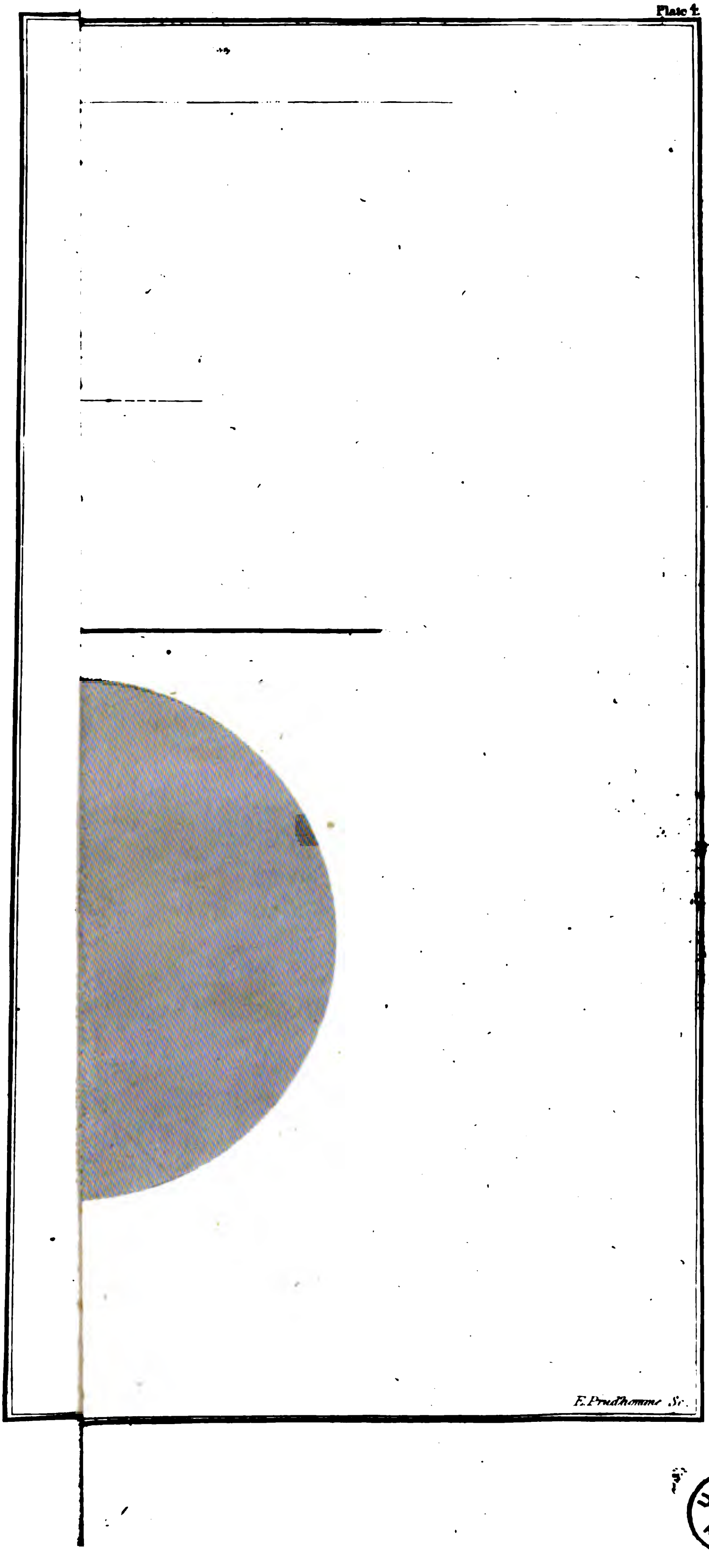
Suppose the sphere to be circumscribed by a tangent cylinder of rays. The axis of the cylinder is the ray of light passing through the centre of the sphere; the curve of contact is a great circle whose plane is perpendicular to the axis of the cylinder, that is, to the direction of the light in space. This circle of contact is the curve of shade.

Since the axis of the tangent cylinder of rays is oblique to both planes of projection, the plane of the circle of shade which is perpendicular to it, is also oblique to both the planes of projection, and consequently its projections on these planes will be ellipses. —(Des. Geom. 180.)

To find the projection of the circle of shade on the horizontal plane.

The horizontal projection of that diameter of the circle of shade which in space is parallel to the horizontal plane, is the transverse axis of the ellipse into which the circle of shade is projected. The projection of that diameter which makes the greatest angle with the horizontal plane, or which is perpendicular to the parallel diameter, is the conjugate axis.—(Des. Geom. 180.)

Plate 4.



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The ray of light through the centre of the sphere being perpendicular to the plane of shade, is also perpendicular to that diameter of the circle of shade which is parallel to the horizontal plane, and therefore the projections of this diameter and the ray of light on the horizontal plane, are at right angles to each other. Hence the diameter CAD, drawn through the centre A, and perpendicular to AB, the horizontal projection of the ray through the centre, is the transverse axis of the ellipse into which the circle of shade is projected.

Through the centre of the sphere suppose a plane of rays to be drawn perpendicular to the horizontal plane. Its horizontal trace is EAB. This plane intersects the surface of the sphere in a great circle, the cylinder of rays in two elements tangent to this circle, and the plane of the circle of shade in a diameter passing through the points of contact. The projection of this diameter is the conjugate axis of the ellipse.

To find the conjugate axis:

Let this plane be revolved about a vertical axis passing through the point A, till it becomes parallel to the vertical plane of projection. In the revolution, the point B describes in the horizontal plane the arc BB′, and the point (A, A′) being in the axis, remains fixed. Projecting B′ into the ground line, and drawing A′B″, we have the vertical projection of the revolved ray. But the great circle cut out of the sphere is, when revolved parallel to the vertical plane, projected into the circle representing the vertical projection of the sphere, and the elements cut from the cylinder of rays are projected into lines parallel to A′B″.

Drawing therefore the tangents F′H′ and G′b′ parallel to A′B″, we determine, in the revolved position of the plane of rays, the extremities of that diameter of the circle of shade whose projection is the conjugate axis.

When the plane of rays is parallel to the vertical plane, the point F′ is horizontally projected at F, and the point G′ at G. In the counter-revolution the points (F, F′) (G, G′) describe the horizontal arcs (F*f*, F′*f*′), (G*g*, G′*g*′), and *gf* becomes the horizontal projection of the diameter, or conjugate axis of the ellipse.. Having found the two axes of the ellipse, let it be described.

It is plain that (*f*, *f*′) is the highest point of the curve of shade, and (*g*, *g*′) the lowest; therefore the tangents to the curve of shade at these points are horizontal, and their vertical projections parallel to the ground line.

Since the points which are horizontally projected at C and D are contained in the horizontal plane through the centre of the sphere, they will be vertically projected in its trace N′B‴, at the points C′ and D′.

The points which are horizontally projected at the intersections of the ellipse C*g*D*f*, with the line NB′, are vertically projected in the circumference of the circle representing the vertical projection of the sphere. We have thus found six points of the ellipse into which the circle of shade is projected on the vertical plane. And since the tangents passing through the highest and the lowest points *f*′ and *g*′ are parallel to the ground line, it follows that the diameter passing through these points is conjugate with the diameter D′C′; therefore let the curve be described (29).

The axes of the ellipse into which the circle of shade is projected on the vertical plane can, however, be found by a construction similar to that used for the horizontal projection.

Through A′ draw a diameter perpendicular to A′B, the vertical projection of the ray. This will be the transverse axis of the ellipse. Through the centre of the sphere conceive a plane of rays to be drawn perpendicular to the vertical plane—A′B is its vertical trace. Let this plane be revolved about an axis passing through the centre of the sphere and perpendicular to the vertical plane, until it becomes parallel to the horizontal plane. The point B describes in the vertical plane the arc BB‴; and since (A, A′) remains fixed, AB″ is the horizontal projection of the revolved ray passing through the centre of the sphere. The lines cH, hb' drawn parallel to the revolved ray and tangent to the circle NDC, determine (c, c'), (h, h'), the extremities of the diameter, in its revolved position, whose vertical projection is the conjugate axis of the ellipse. In the counter revolution, the points (c, c') and (h, h') describe the arcs $(c\,d, c'd')$ and $(h\,k, h'k')$: and $k'd'$ is the conjugate axis of the ellipse into which the circle of shade is projected on the vertical plane.

In horizontal projection, we see only that part of the shade which lies above the horizontal plane N′B‴; the part of the curve of shade lying below this plane is dotted. In vertical projection, we see that part of the shade which lies in front of the plane NB′.

To find the shadow on the horizontal plane:

The curve of shadow on the horizontal plane is the intersection of the horizontal plane with the surface of the cylinder of rays tangent to the sphere. This curve will be an ellipse, unless the circle of shade be parallel to the horizontal plane. When the plane of rays AB was revolved parallel to the vertical plane, we drew F′H′ and G′b' parallel to the revolved ray A′B″. The tangent (G′b', Gb) pierces the horizontal plane at b. In the counter revolution, the point b describes in the horizontal plane the arc $b\,b'''$; then b'''B is the shadow cast by the radius of the circle of shade which passes through the lowest point—hence it is the semi-transverse axis of the ellipse of shadow. The conjugate axis of this ellipse is the shadow cast by the horizontal diameter of the circle of shade. It passes through B, is perpendicular to b''B, and equal to CD. The shadow cast on the vertical plane is found in a manner entirely similar.

OF BRILLIANT POINTS.

35. When a ray of light falls upon a surface which turns it from its course and gives it another direction, the ray is said to be reflected. The ray, as it falls upon the surface, is called the incident ray, and after it leaves the surface, the reflected ray. The point at which the reflection takes place is called the point of incidence. It is ascertained by experiment,

1°. That the plane of the incident and reflected rays is always normal to the surface at the point of incidence.

2°. That at the point of incidence, the incident and reflected rays make equal angles with the tangent plane or normal line to the surface.

If therefore, we suppose a single luminous point, and the light emanating from it to fall upon any surface and to be reflected to the eye, the point at which the reflection takes place is called the brilliant point. The brilliant point of a surface is, then, the point at which a ray of light and a line drawn to the eye make equal angles with the tangent plane or normal line—the plane of the two lines being normal to the surface.

36. The rays of light being parallel, and the place of the eye at an infinite distance, the brilliant point of any surface is thus found:

Through any point in space draw a ray of light and a line to the eye, and bisect the angle included between them. Then draw a tangent plane to the surface and perpendicular to the bisecting line—the point of contact is the brilliant point. For, let AC (Fig. *n*) be the line drawn to the eye, AB the ray of light, AD the bisecting line, and P the point of contact of a plane passed perpendicular to AD, and tangent to the surface. Through P let there be drawn a ray of light PG which will be parallel to AB, the line PE to the eye which will be parallel to AC, and PF the normal line to the surface at the point P.

Since the normal PF is perpendicular to the tangent plane, it is parallel to the bisecting line AD, for AD is perpendicular to the tangent plane by construction. But the line AD is in the plane of the lines BA, AC, and bisects the angle BAC: therefore the parallel PF lies in the plane of the lines GP, PE, and bisects the angle GPE. Hence P is the brilliant point.

36. To apply these principles in finding the brilliant point on the surface of a sphere.

Suppose the eye to be in a line perpendicular to the vertical plane, and at an infinite distance from it.

Through (A, A′), the centre of the sphere, suppose a ray of light to be drawn, and also a line to the eye. For the purpose of bisecting the angle included between these lines, let their plane be revolved about (AP, A′) the line drawn to the eye, until it becomes parallel to the horizontal plane. Any point of the ray, as (E, E′), will describe the arc (E*e*, E′*e*′); and A*e* and AP are the horizontal projections of the lines when revolved parallel to the horizontal plane. Bisect then, the angle PA*e* by the line A*q*: and from any point of the axis, as P, draw the line P*qe*. After the counter revolution, the point *e* is horizontally projected at E; and since P remains fixed, P*qe* is horizontally projected in PE; the point *q* of the bisecting line, is projected at *q*′, and A*q*′ is the horizontal projection of the bisecting. Its vertical projection is A′E′.

The plane drawn perpendicular to this bisecting line, and tangent to the sphere, touches the surface at the point where the bisecting line pierces it—that is, at (a', a''). Consequently, (a', a'') is the brilliant point. A similar construction would determine the brilliant point, if the eye were taken in a line perpendicular to the horizontal plane.

PROBLEM X.

Having given an ellipsoid in space, and the direction of the light, it is required to find the curve of shade, and the shadow cast on the horizontal plane.

37. Let the horizontal plane be taken perpendicular to the axis of the surface. Let A (Pl. 6. Fig. 1) be the horizontal projection of the axis, and A′B its vertical projection. Let us suppose the ellipsoid to be circumscribed by a tangent cylinder of rays. The axis of the cylinder is a ray of light passing through the centre of the ellipsoid and piercing the horizontal plane at C. Through this axis let a plane of rays be passed perpendicular to the horizontal plane—ACD is its trace.

Since this plane is a plane of rays, and divides the ellipsoid into two equal and symmetrical parts, the parts of the curve of shade lying on either side are equal and symmetrical. Hence, both parts are projected on the meridian plane AC, into the same right line. But, since the contact of the ellipsoid and surface of the cylinder is an ellipse whose plane passes through the centre of the ellipsoid, the curve of shade is projected into a right line passing through the centre, which line is the intersection of the plane of the curve of shade with the meridian plane ACD. The plane ACD intersects the surface of the ellipsoid in a meridian curve, and the sur-

Plate 5.

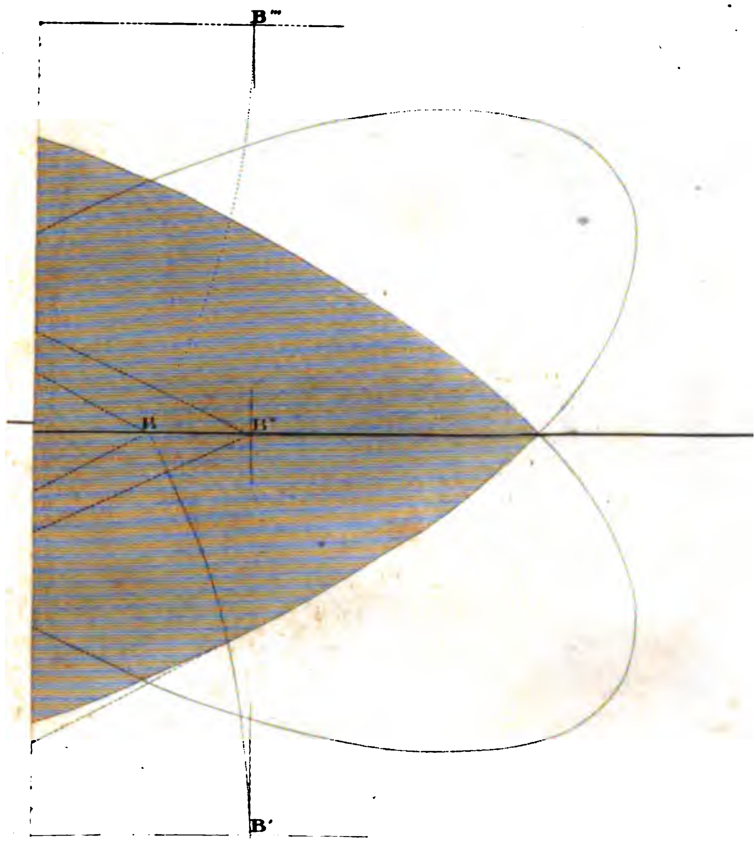
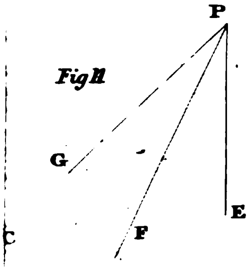


Fig II



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face of the cylinder of rays in two elements that are tangent to it. The points of tangency are the highest and lowest points of the curve of shade. The line in which the plane of shade intersects the meridian plane AC, also passes through these points.

Let the ellipsoid be now projected on a vertical plane, parallel to the meridian plane ACD. Assuming E′D′ for the new ground line, the centre of the ellipsoid will be projected at F, in the perpendicular AF to the ground line E′D′, and at a distance from it, equal to the distance of the centre of the ellipsoid from the horizontal plane. The ray of light through the centre will be vertically projected in FC′. The ellipse described about the centre F, and equal to a meridian curve of the surface, represents the vertical projection of the ellipsoid.

Having described this ellipse, draw the two tangents d''D′ and e''E′ parallel to FC′, the vertical projection of the ray. The points of contact d'', e'' are the vertical projections of the highest and lowest points of the curve of shade; and d''Fe'' is the vertical projection of the curve of shade, since the plane of shade is perpendicular to the new vertical plane. The horizontal projections of the highest and lowest points are d and e.

Let us suppose a system of horizontal planes to intersect the ellipsoid between the highest and lowest points of the curve of shade. Each of such secant planes, being perpendicular to the axis, will intersect the surface of the ellipsoid in a horizontal circle, and the plane of shade in a horizontal line perpendicular to the meridian plane ACD, and consequently, to the new vertical plane.

Let $f'n$ be the vertical trace, on the new vertical plane, of one of the secant planes. The line $f'n$ is the vertical projection of the circle in which the plane intersects the surface of the ellipsoid, and h'' of the line in which it intersects the plane of shade. Projecting the circle and line on the horizontal plane, the points h and k, in which they intersect, are two points in the horizontal projection of the curve of shade. In a similar manner any number of points may be determined. The horizontal plane through the centre of the ellipsoid, intersects the plane of shade in a line whose horizontal projection st is the transverse axis of the ellipse into which the curve of shade is projected, and ed, the projection of the diameter joining the highest and lowest points, is the conjugate axis.

The projection of the curve of shade on the primitive vertical plane, is found by projecting the horizontal circles, and drawing perpendiculars to the ground line through points of the horizontal projection of the curve. Thus, projecting the horizontal circles passing through the highest and lowest points, and drawing perpendiculars to the ground line from d and e, determines d' and e', the vertical projections of the highest and lowest points. The vertical projection of the circle $f'h''n$, is $h'k'$, and h', k' are the vertical projections of the points of shade that are horizontally projected at h and k. The diameters $d'e'$ and $t's'$ are conjugate.

The part of the surface which is in the shade, and in front of the meridian plane gAp, is shaded in vertical projection. The part in the shade and above the horizontal plane passing through the centre of the ellipsoid, is shaded in horizontal projection.

To find the shadow on the horizontal plane:

If we suppose the lines in which the horizontal secant planes intersect the plane of shade, to cast shadows on the horizontal plane, the shadows cast will be parallel to the lines themselves. The line in which the horizontal plane fn intersects the plane of shade, casts the shadow $h'''k'''$, and the points h''' and k''' in which it is intersected by hh''' and kk''', drawn parallel to the horizontal projection of the light, are points in the shadow on the horizontal plane. The highest and lowest points cast their shadows at D and E.

If through the highest and lowest points of the curve of shade, two planes be drawn tangent to the cylinder of rays circumscribing the ellipsoid, their traces on the horizontal plane will be tangent to the ellipse of shadow; but since the planes are perpendicular to the vertical plane EAD, their traces will be perpendicular to ED, at the points E and D. Hence, ED is an axis of the ellipse of shadow. The other axis passes through the middle point C, and is the shadow cast by the diameter $(st, s't')$.

PROBLEM XI.

Having given the plan and elevation of a niche, and the direction of the light, it is required to find the shadow which the niche casts upon itself.

38. A niche is a recess in the wall of a building. It is generally composed of a semi-cylinder and the quadrant of a sphere, having the same radius as the base of the cylinder. The quadrant of the sphere rests on the upper base of the semi-cylinder, forming the upper part of the niche. The quadrant and semi-cylinder are tangent to each other in the semicircle which separates the cylindrical from the spherical surface.

Let AB (Pl. 7. Fig. 1) be the intersection of the face of a vertical wall with a horizontal plane, and the semicircle ACB the plan of the niche. The rectangle A''B' is the elevation of the cylindrical part, and the semicircle A″F′B″ of the spherical part of the niche—A″B″ is the vertical projection of the semicircle that separates the cylindrical from the spherical surface.

The lines of the niche which cast lines of shadow on the surface, are,

1°. The element (A, A′A″).

2°. The semicircle A″F′B″.

A part of the shadow cast falls on the base of the niche, a part on the cylindrical, and a part on the spherical surface.

To find the shadow on the base and cylindrical surface.

Through the element (A, A′A″) conceive a plane of rays to be passed. Its horizontal trace AC is parallel to the horizontal projection of the rays of light, and the plane intersects the cylindrical surface in a second element (C, C′*a*), opposite the source of light, and on this element the shadow falls. The line A″*a*, drawn through A″ parallel to the vertical projection of the rays of light, limits the shadow on the element: and drawing through C′ the line C′*b*, parallel to the vertical projection of the ray of light, determines the point *b*, which casts a shadow at C. The point C is common both to the cylindrical surface and to the base of the niche. Hence, the part A′*b* of the element (A, A′A″) casts the shadow AC on the base of the niche, and the part *b*A″, the shadow C′*a* on the cylindrical surface. Above the point *a*, the shadow on the cylindrical surface is cast by the semicircle (AB, A″F′B″).

Through any point of the semicircle, as (F, F′), conceive a vertical plane of rays to be passed. Its horizontal trace F*f* is parallel to the horizontal projection of the rays of light, and the plane intersects the surface of the cylinder in the element (*f*, *ff*″). The point *f*″, in which the vertical projection of the ray drawn through

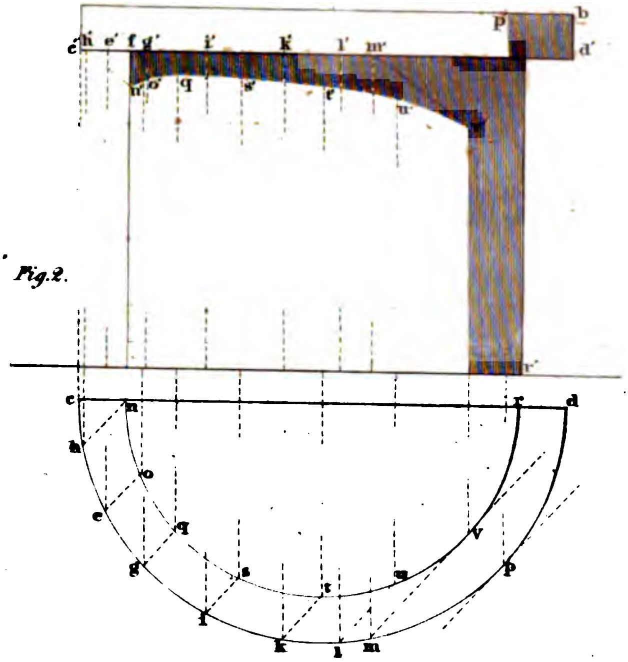

Fig. 2.

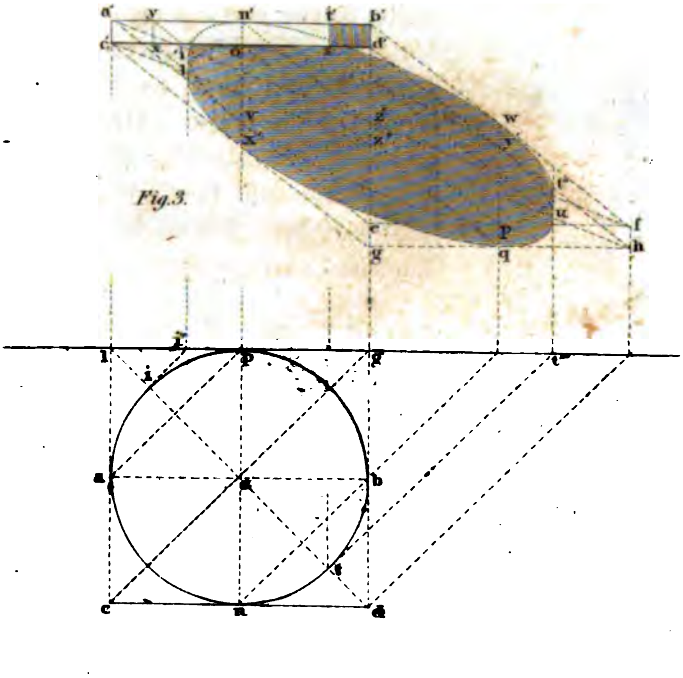

Fig. 3.

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F meets (ff''), is the vertical projection of the shadow cast by the point (F,F'). In a similar manner, we may determine other points of shadow on the cylindrical surface.

Above the line A″B″ the shadow will fall on the spherical surface. Before finding this shadow we will demonstrate, that when a cylinder intersects a sphere, the curve in which it enters on the one side, is equal to the curve in which it leaves the sphere on the other.

For, the parts of the elements of the cylinder intercepted between the points at which they enter, and the points at which they leave the sphere, are parallel chords of the sphere. Conceive a plane to be passed through the middle point of one of these chords and perpendicular to it. Such plane passes through the centre of the sphere and bisects all the other parallel chords. Hence, the curves in which the cylinder enters and leaves the sphere, are symmetrical with respect to this plane, and are consequently, equal. Therefore, if one of them is a circle, the other will be an equal circle; and hence, when the surface of a cylinder intersects that of a sphere in a great circle, it will at the same time intersect it in a second great circle, and the planes of these circles will intersect each other in a diameter of the sphere. The two elements of the cylinder, passing through the extremities of this common diameter, are perpendicular to it, since they are tangent to the sphere.

Let us suppose an entire hemisphere to be described on the diameter (AB, A″B″), having the plane of its great circle vertical. Through this circle suppose a cylinder of rays to be passed. Half the surface of this cylinder intersects the surface of the hemisphere in a semicircle, which is the shadow that the semicircle (AB, A″F″B″) would cast on the hemisphere.

The plane of the circle casting the shadow, and the plane of the circle of shadow, intersect each other in a line passing through the centre of the hemisphere and perpendicular to the elements of the cylinder of rays, that is, perpendicular to the direction of the light. But, since this diameter is a line of the vertical plane AB, it is parallel to the vertical plane of projection—hence, its vertical projection is at right angles to the vertical projection of the rays of light (Des. Geom. 49). Therefore, if through D', we draw the line $D'k'D''$ perpendicular to the projection of the ray, we determine Ik', the vertical projection of the diameter in which the plane of the circle casting the shadow intersects the plane of the circle of shadow. The semicircle $IA''n'k'$ casts the semicircle of shadow on the hemisphere, the vertical projection of which shadow is an ellipse, whose transverse axis is Ik' (Des. Geom. 180).

Through (D, D') let a plane of rays be passed perpendicular to the face of the niche, and let us suppose for a moment the vertical plane of projection to be moved forward to coincide with this face. The point D' will then be directly over D, and $n'D'p$ will be the trace of the plane of rays.

Let us now assume an auxiliary plane of projection, parallel to the plane whose trace is $n'p$, and at a given distance from it. Draw $n''p'$ parallel to $n'p$, to represent the trace of the auxiliary plane, and let it be borne in mind that this trace, as well as $n'p$, is in the plane of the face of the niche. The semicircle in which the plane of rays $n'p$ intersects the hemisphere, is projected on the auxiliary plane in the semicircle $n''k'p'$. The projections of the ray of light passing through the point (n, n'), are nk and $n'D'$. This ray is projected on the auxiliary plane by projecting n' into the trace at n'', and projecting any other point of the ray, as (k, D') (Des. Geom. 14): this is done by laying off $D''k'$ equal to Dk. The line $n''k'l$ is the projection of the ray on the auxiliary plane, and the point l, in which it meets the semicircle $n''lp'$, is the projection of a point of the curve of shadow. But since $D'k'$, a line in the plane of the circle of shadow, is perpendicular to the plane of rays whose trace is $n'p$, and consequently to the auxiliary plane, it follows, that the plane of shadow is perpendicular to these planes; hence, the projection of the shadow on either of them is a right line (Des. Geom. 82). But D'' is the projection of h', a point of the curve of shadow, and a second point is projected at l; hence, $D''l$ is the projection of the shadow on the auxiliary plane. This line is also the projection of the intersection of the plane of rays, whose trace is $n'p$, with the plane of the circle of shadow.

In space, therefore, the line whose projection on the auxiliary plane is $D''l$, is perpendicular to the diameter lh': hence, $D'l'$ its projection on the vertical plane, is the semi-conjugate axis of the ellipse into which the circle of shadow is projected (Des. Geom. 180). Let the ellipse be described. From h' to the point q', where the ellipse intersects at $A''B''$, the shadow falls on the surface of the sphere. From q' the shadow falls on the cylindrical surface. The arc $h'E$ casts the shadow on the spherical, and the arc $A''E$ the shadow on the cylindrical surface.

Had the quadrant of the sphere below the plane $A''B''$ been permitted to intercept the light, the shadow $af''q'$, instead of being on the cylinder, would have been the shadow $q'l'I$ on the surface of the sphere. We have also dotted the line of shadow $ae'vq'$, which the front circle of the hemisphere would cast upon the vertical cylinder, if the sphere did not intercept the rays of light.

The part of the surface of the niche lying between the lines $A'A''Eh'$ and the line of shadow, is shaded in vertical projection.

It is not necessary to find the semi-conjugate axis of the ellipse into which the circle of shadow is projected. We may find points of the curve, and describe it through them.

Let $i's'$ be the vertical trace of a plane of rays perpendicular to the face of the niche. This plane intersects the hemisphere in a semicircle which is projected on the auxiliary plane in the semicircle described with the radius $D''i''$. Draw $i''s''$ parallel to $n''l$. The point s'', where it meets the semi-circle and the line $D''l$, is the projection on the auxiliary plane of a point of the circle of shadow. This point is projected on the vertical plane of projection at s', and on the horizontal plane at s, by making ms equal to $m's''$. In a similar manner other points of the curve may be found.

The point h' is projected on the horizontal plane at h, and the point q' at q: therefore, hsq is the horizontal projection of the shadow which falls on the spherical part of the niche. The shadow on the cylindrical part is horizontally projected in the arc Ceq.

The point (q, q'), at which the shadow passes from the cylindrical to the spherical part of the niche, can be found by a direct construction.

The point in space, is the one in which the intersection of the upper base of the cylinder with the plane of shadow, meets the circle of the upper base. But $h'l$ is the trace of the plane of shadow on the face of the niche; and (s, s') is a point of this plane. Therefore, we have one trace of an oblique plane, and a point of the plane, to find its other trace (Des. Geom. 43). Draw $s'u'$ parallel to $h'D'$; its horizontal projection is su, and (u, u') is the point in which it pierces the horizontal plane A″B″: therefore, (u, u') is a point in the trace of the plane of shadow. But the trace passes through (D, D′); hence Duq is the horizontal projection of the required trace. But q' is the vertical projection of q; hence (q, q') is the point at which the shadow passes from the cylindrical to the spherical surface.

PROBLEM XII.

To find the curve shade on the surface of a torus.

39. Let $abcg$ (Pl. 7, Fig. 2) be a rectangle, having the semicircles $aA'b$ and $cB'g$ described on its vertical sides.

If the figure $aA'bcB'g$ be revolved about a vertical axis (C, C″O), it will generate a solid, called a torus. This solid is used in some of the orders of architecture in forming the bases and capitals of the columns.

Before finding the curve of shade, we will demonstrate, that if a surface of revolution be intersected by a meridian plane, and a line be drawn tangent to the meridian curve and parallel to the projection of the rays of light on the meridian plane, the point of contact will be a point of the curve of shade.

For, every plane of rays, tangent to a surface, touches it in a point of the curve of shade. But since the surface is one of revolution, such plane is perpendicular to the meridian plane passing through the point of contact (Des. Geom. 105). Therefore, its trace will not only be tangent to the meridian curve, but also parallel to the projection of the rays of light, since the rays are projected by perpendicular planes. Hence, *the tangent to a meridian curve, drawn parallel to the projection of the rays of light on its plane, determines a point of shade.*

Let the circle ATBH represent the horizontal projection of the surface, $gB'cbA'a$ its vertical projection, and (A, A′) the ray of light. Through any point of the axis of the surface, as (C, C′), suppose a ray of light to be drawn, CD, C′D′ are its projections. Through this ray let a meridian plane be passed, and then revolved about the axis of the surface until it becomes parallel to the vertical plane of projection. The point (C, C′) being in the axis, remains fixed, and any point of the ray, as (D, D′), describes a horizontal arc (Dd, D′d'); and the vertical projection of the ray, from its revolved position, is $C'd'$. The section of the surface, when revolved parallel to the vertical plane, is vertically projected in the meridian line $gB'cbA'a$.

If now, two lines be drawn tangent to the semicircles and parallel to C′d', the revolved ray, they will determine the points (k, k') and (i, i'): these are the highest and lowest points of the curve of shade, in their revolved position. In the counter revolution of the meridian plane, they describe horizontal arcs, and when the counter revolution is completed, are horizontally projected at h and l, and vertically at h' and l'.

The lines drawn parallel to C′D′, the vertical projection of the ray, and tangent to the semicircles, determine the points (q, q') and (p, p') at which the curve of shade intersects the meridian plane AB.

The points (T, t') and (H, t'') of the curve of shade are found by passing two planes of rays tangent to the surface and perpendicular to the horizontal plane. The planes will touch the surface in the circumference of the circle whose vertical projection is A′B′. The points of tangency are also found in a meridian plane passed perpendicular to the plane DCh.

To find other points of the curve of shade:

Draw any meridian plane, as nCm, and project the ray of light upon it. The point (C, C') is its own projection, and (D, D') is projected at (F, F'); therefore, $(CF, C'F')$ is the projection of the ray on the meridian plane nCm. Let the meridian plane be revolved till it becomes parallel to the vertical plane of projection. The line $(CF, C'F')$ will then be vertically projected in the line $C'f$. Draw two lines parallel to $C'f$ and tangent to the semicircles; their points of contact (r, r') and (v, v') are points of the curve of shade in their revolved position. In the counter revolution these points describe horizontal arcs, and when it is completed are horizontally projected at n and m, and vertically at n' and m'. Any number of points of the curve of shade may be found in a similar manner. The curve $HqlTphm$ is the horizontal, and $q'l'n't'p'h'm'$ the vertical projection of the curve of shade.

PROBLEM XIII.

Having given a surface of revolution, convex towards the axis, and the direction of the light, it is required to find the shadow which the upper circle casts upon the surface, and also the brilliant point.

40. Let DRN (Pl. 8, Fig. 1) be the horizontal, and $D'L$ the vertical projection of a cylinder or pedestal, on which the surface rests.

Take two-thirds of the radius AD, and lay it off from A' to v, on the vertical line $A'B$. Through v draw the horizontal line vu, which meets the vertical line $D'u$ at u. With u as a centre, and radius uD', describe the quadrant D′H′X′. Then lay off vB equal to one-third of the radius A′D′, and draw Bl' parallel to A′D′, and equal to two-thirds of it. With y as a centre, and a radius equal to yX′ or yl', describe the quadrant X′bl'. The two quadrants will have a common tangent line at the point X′, which will be vertical. Now, if the curve D′H′X′l' be revolved about the vertical axis (A, A′B) it will generate a surface of revolution, convex towards the axis, and concave outwards. The circle whose radius is X′v is called the circle of the gorge.

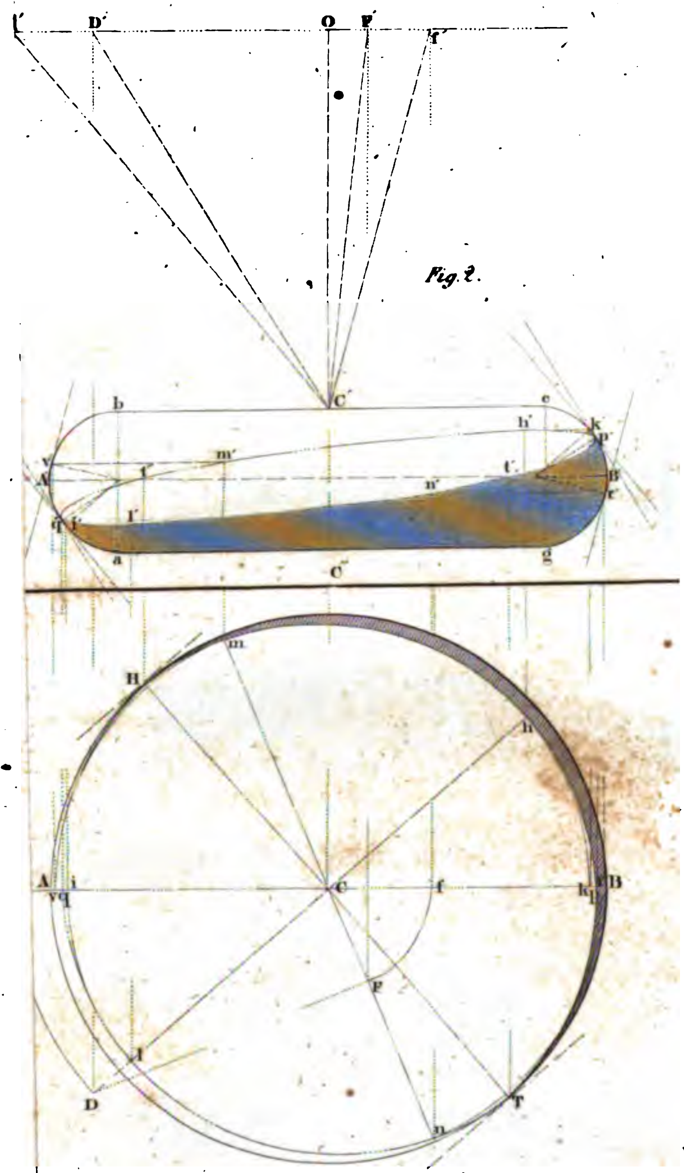
Let us also suppose a cylinder, having the radius of its base equal to the radius of the upper circle of the surface, to be placed on the surface.

It is required to find the curve of shadow which the upper circle (laQS, $l'z'$) casts upon the surface, under the supposition that the light is not intercepted by the surface. Admitting this supposition, it follows, that each point of the upper circle casting a shadow will, in general, cast two points of shadow; one where the ray through it enters the surface, and the other where it leaves the surface.

Through the upper circle of the surface suppose a cylinder of rays to be passed. The ray (AC, BC′) is the axis of this cylinder, and the curve in which it intersects the surface of revolution is the curve of shadow required.

Through the axis of the cylinder of rays and the axis of the surface, suppose a meridian plane to be passed. Its horizontal trace is AC, and it cuts the upper circle of the surface in two points, one of which (a, a') casts shadows on the meridian curves. Let this plane be revolved parallel to the vertical plane of projection. After it is so revolved, the ray of light will be vertically projected in Bc', the meridian curves in the curves representing the vertical projection of the surface, and the point (a, a') at l'.

Fig. 2.



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Through l' draw $l'bd$ parallel to the revolved ray Bc'; the points b and d in which it intersects the meridian curves, are the highest and lowest points of the curve of shadow, in their revolved position. In the counter revolution the points b and d describe arcs of horizontal circles, and when the revolution is completed, are horizontally projected at h and g, and vertically at h' and g'. These points may also be found by drawing through v, the point in which the revolved ray $l'bd$ meets the axis, the line $g'vh'$ parallel to BC', the vertical projection of the ray, and noting its intersection with the horizontal lines dg' and bh'; and then projecting the points g' and h' into the horizontal plane in the line aAC.

To find points of the curve between the highest and lowest points, we intersect by horizontal planes. Each horizontal plane will intersect the surface of revolution in a circle, and the surface of the cylinder of rays in a circle equal to the upper circle of the surface: the points in which these circles intersect are points of the curve of shadow.

Let H'C' be the trace of an auxiliary horizontal plane. This plane cuts the axis of the cylinder of rays in the point (C, C'). With C as a centre, and a radius equal to Bz', describe the arc pm; this will be the horizontal projection of an arc of the circle in which the auxiliary horizontal plane intersects the surface of the cylinder of rays. Projecting H' into the line DA, we have AH for the radius of the circle in which the same plane intersects the surface of revolution. Describing that circle, and noting the points in which it intersects the circle described with the centre C, we find m and p, the horizontal projections of two points of the curve of shadow. These points are vertically projected in the vertical trace of the auxiliary plane, at m' and p'. Similar constructions determine the points (i, i'), (e, e'), (o, o'), and (n, n'). The points (i, i') and (e, e') are in the circle of the gorge.

That part of the curve is made full, in vertical projection, which is in front of the meridian plane DAz; and the part of the surface which is above the curve of shadow is shaded. The elements of shade on the pedestal are (N, N') and (R, R'); and on the upper cylinder (Q, Q') and (S, S').

41. To find the brilliant point:

Suppose the eye to be situated in a perpendicular to the vertical plane, and at an infinite distance from it.

Through any point of the axis, as (A, A'), suppose a ray of light to be drawn, and also a line to the eye, and the angle contained between them to be bisected as in Art. 36. The bisecting line is (AK, A'K'). It is now required to pass a plane perpendicular to this line and tangent to the surface; the point of contact will be the brilliant point.

If we suppose the tangent plane to be drawn, its trace on the meridian plane passing through the bisecting line (AK, A'K') will be perpendicular to the bisecting line, and tangent to the meridian curve. Let this meridian plane be revolved parallel to the vertical plane of projection. The bisecting line will then be vertically projected in the line A'k', and the meridian curve in the curve D'H'l'. Let H'l be drawn perpendicular to A'k', and tangent to the curve D'H'l'; the point of contact (H, H') is the revolved position of the brilliant point. Draw the normal H'V perpendicular to the tangent, or parallel to A'k'. In the counter revolution, V being in the axis remains fixed, and H' describes the arc of a horizontal circle. After the counter revolution, the bisecting line is vertically projected in A'K', and the normal VH' in

VP′, parallel to A′K′. Hence P′, where the horizontal line H′P′ intersects VP′, is the vertical projection of the brilliant point. Its horizontal projection is at P, in the horizontal trace of the meridian plane AK.

The construction here given, is general for all surfaces of revolution. If the eye is supposed to be in a line perpendicular to the horizontal plane, the brilliant point is easily found; for we bisect the angle as before, and draw a tangent plane perpendicular to the bisecting line.

It is plain that a second line can be drawn perpendicular to *k*′A′, produced on the other side of A′, which shall be tangent to the meridian curve *z*′*d*. A second tangent plane can therefore be drawn perpendicular to the bisecting line, and the point of contact will answer the mathematical conditions of a brilliant point. The point, however, will be on that part of the surface which is not seen by the eye.

PROBLEM XIV.

Having given a surface of revolution and the direction of the light, it is required to find the curve of shade.

42. Fig. 2, Pl. 8 represents the projections of a surface of revolution generated as in the last problem.

Every ray of light that is tangent to the surface of revolution is an element of the tangent cylinder of rays which determines the curve of shade; therefore, every point at which a ray of light is tangent to the surface is a point of the curve of shade.

Through the axis of the surface let a meridian plane of rays be drawn—PB is its horizontal trace. Let this plane be revolved until it becomes parallel to the vertical plane of projection. The ray of light through (A, A′)

will then be vertically projected in the line $A'b'$, and the meridian curves, in the curves which represent the vertical projection of the surface. Let the two tangents $a'd$ and ge' be drawn parallel to the revolved ray $A'b'$; the points of contact (a, a') and (e, e') are the highest and lowest points of the curve of shade in their revolved position. After the counter revolution, these points are horizontally projected at c and f, and vertically at c' and f'. The lines gf' and dc' are parallel to $A'B'$ the vertical projection of the ray.

To find other points of the curve of shade, we use auxiliary tangent surfaces.

Draw any line, as $E'k'$, between the highest and lowest points. At E' or k', draw a tangent, as $E'C$, to the meridian curve. Conceive the meridian plane and the right-angled triangle $E'CO$ to be revolved about the axis of the surface. The meridian curve generates the surface of revolution, and the line $E'C$ the surface of a right cone tangent to it in a circle whose vertical projection is $E'k'$, and horizontal projection Emn. If now, two tangent planes of rays be passed to this cone, they will be also tangent to the surface of revolution at two points in the circumference of the circle $(Emn, E'k')$: these points are points of the curve of shade.

Through (A, C), the vertex of the cone, let a ray of light be drawn. This ray pierces the plane of the cone's base at (D, D'). Through (D, D') let two lines be drawn tangent to the base of the cone—these tangents are the traces of the two planes of rays that are tangent to the cone, and Dm and Dn are their horizontal projections. Projecting the points m and n into the vertical plane at m' and n', we determine the two points (m, m') and (n, n') of the curve of shade. In a similar manner, other points of the curve of shade may be found.

If the circle of contact (E*mn*, E′*k*′) be taken nearer the circle of the gorge, the vertex of the cone will be farther from the horizontal plane, and the elements will be nearer vertical. And when the circle of the gorge is assumed for the circle of contact, the auxiliary tangent surface will be a vertical cylinder, having a common axis with the surface of revolution. If two planes of rays be drawn tangent to this cylinder, the elements of contact will pierce the horizontal plane at *i* and *l*, the opposite extremities of a diameter perpendicular to the horizontal projection of the ray of light. This cylinder determines the two points (*l*, *l*′) and (*i*, *i*′) of the curve of shade.

There is yet a third method of finding points of the curve of shade: it is by means of auxiliary tangent spheres.

Assume any line, as *p*′*r*, for the vertical projection of the circle of contact of a sphere and the surface of revolution. At *p*′ draw a tangent to the meridian curve, and *p*′*q* perpendicular to it. The point *q*, where the perpendicular meets the axis, is the vertical projection of the centre of the sphere, and *qp*′ is its radius. Let us now suppose this sphere to be circumscribed by a tangent cylinder of rays. This cylinder will touch the sphere in a great circle, the plane of which will be perpendicular to the direction of the light. Therefore, the trace of the plane of the circle of contact of the sphere and cylinder of rays, on the meridian plane EA*b*, or on the vertical plane of projection, will be perpendicular to the projection of the ray of light on either of these planes (Des. Geom. 49). Hence, *q s*′, drawn through *q* perpendicular to A′B′, is the vertical projection of the line in which the plane of the circle of contact of the sphere and cylinder intersects the meridian plane EA*b*.

But the plane of this circle of contact intersects the plane of the circle of contact of the sphere and surface of revolution in a horizontal line perpendicular to the direction of the light. This horizontal line must pierce the meridian plane EAb in the trace qs', and also in the trace $p'r$; therefore it pierces it at (s, s'). Hence, the line tsv, drawn through s perpendicular to the horizontal projection of the ray, is the horizontal projection of the intersection sought. Projecting p' into the horizontal plane at p, and describing a circle with the radius Ap, we have the horizontal projection of the circle of contact of the sphere and surface of revolution. Projecting the points t and v, in which the line tsv intersects this circle, into the vertical plane at t' and v', and we have the points (t, t') and (v, v') which are common to the surface of revolution, the tangent sphere, and the cylinder of rays.

If through these points planes be drawn tangent to the cylinder of rays, they will be planes of rays, and tangent both to the sphere and surface of revolution. Hence, the points (t, t') and (v, v') are points of the curve of shade.

If we suppose the space within the surface of revolution to be unoccupied, a part of the curve of shade will cast a shadow that will fall on the convex side of the surface.

The ray of light which determines (c, c'), the highest point of shade, being produced, will intersect the opposite meridian curve: the point of intersection is a point of shadow. There are points of the curve of shade, on both sides of the meridian plane PAD, which cast shadows on the surface. As we recede from the meridian plane PAD, on either side, the part of the ray intercepted between the point of shade and the corresponding point of shadow, continually diminishes, and finally becomes nothing, or the point of shadow unites with the point of shade casting it.

At these two points, one on each side of the meridian plane PAD, the ray of light will be tangent to the curve of shade. For, when the point of shadow unites with the point of shade, they become consecutive points of the curve of shade; hence, the ray passing through them is tangent to it (Des. Geom. 65). But if the rays of light at these two points are tangent to the curve of shade, their projections will be tangent to the projections of the curve (Des. Geom. 90). The horizontal projection of the curve of shade being constructed, draw two tangents to it parallel to APD, the horizontal projection of the ray of light. The points of contact are X and *y*, and X*cy* is the horizontal projection of that part of the curve of shade which casts a shadow on the interior of the surface. Projecting the points X and *y* into the vertical projection of the curve, at X′ and *y*′, and drawing lines parallel to the vertical projection of the rays, the lines so drawn will be tangent to the vertical projection of the curve of shade.

Descending along the curve of shade, from the points (X, X′), (*y*, *y*′), the rays of light touch the surface on the concave side, and the points of the curve of shade still cast shadows upon the surface. When we reach those points, one on each side of the meridian plane PAD, at which the point of the curve of shade unites with the point of the curve of shadow, the ray becomes tangent to the curve. Therefore, drawing two other tangents parallel to PD, their points of contact *w* and *z* are the horizontal projections of two points at which the rays of light are tangent to the curve of shade. At these points the curve of shade returns to the convex side of the surface. Projecting these points into the vertical projection of the curve of shade, we find the points w' and z', through which, if the projections of rays be drawn, they will be tangent to the vertical projection of the curve of shade.

That part of the curve of shade whose horizontal projection is ycX lies on the convex side of the surface; the part $Xvlmz$ lies on the concave side of the surface; the part zfw lies on the convex side, and the part $wnity$ on the concave side. The curve is symmetrical with respect to the meridian plane PAD.

The part of the curve of shade which is in front of the meridian plane EAb is made full in vertical projection, and the part of the surface lying above the curve of shade and seen is darkened.

PROBLEM XV.

Having given the position of a surface of revolution and the direction of the light, it is required to find the line which separates the dark from the illuminated part of the surface, and the shadow which is cast on the horizontal plane of projection.

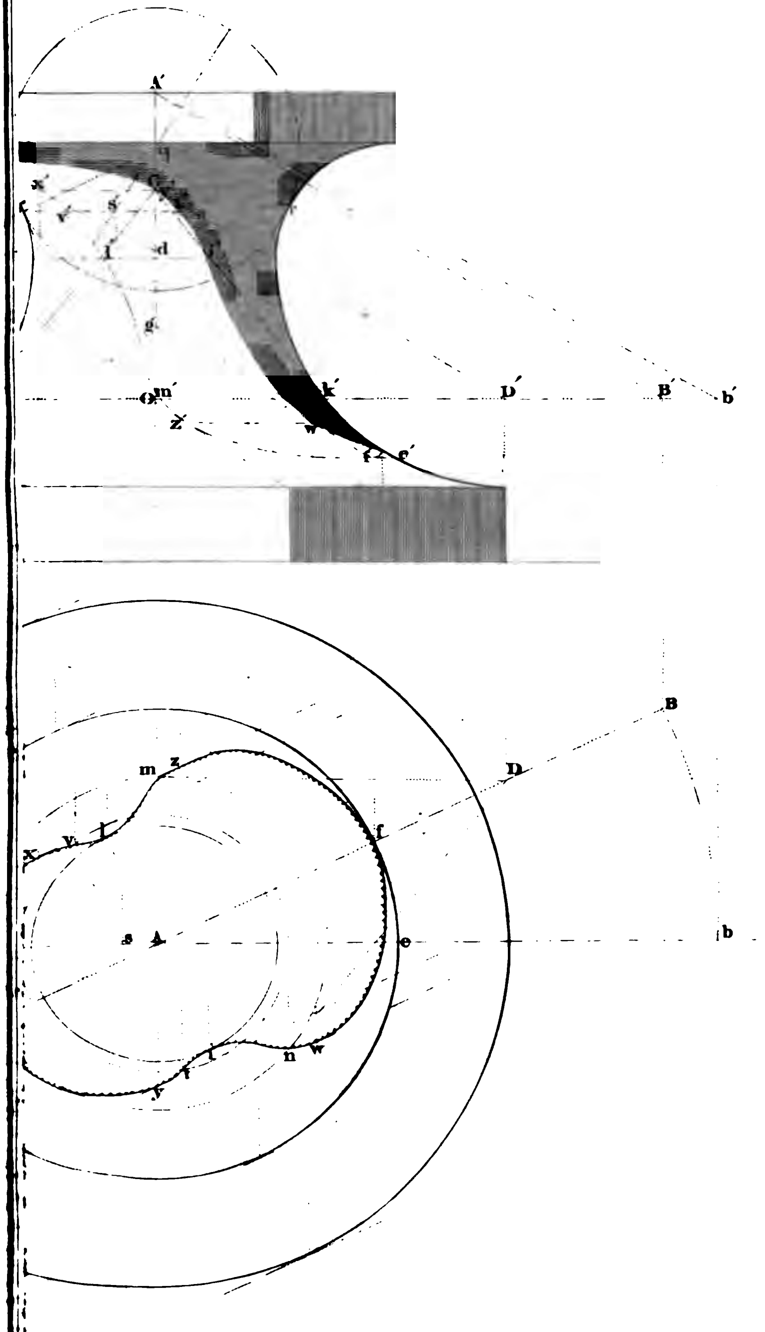
43. Let (A, A′B), (Pl. 9) be the axis of the surface, and let the projections of the surface be made as in the figure.

Find the shadow which the upper horizontal circle CD casts upon the surface, as in Prob. 13, and then find the curve of shade, as in Prob. 14.

The highest and lowest points of the curves of shade and shadow, are in the meridian plane of rays EAB′.

By considering the form of the meridian curves, it is plain, that the highest point of the curve of shade is above the highest point of the curve of shadow, and the

Plate 8.



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lowest point of the curve of shade below the lowest point of the curve of shadow. These curves will therefore intersect each other. This they do at the points (a, a') and (b, b'). Above these points the curve of shadow being below the curve of shade, and on the exterior surface, separates the dark from the illuminated part of the surface.

Below these points, the curve of shade is below the curve of shadow, and separates the dark from the illuminated part of the surface, until it returns to the convex side of the surface at the points (c, c') and (d, d').

It has already been observed, in Prob. 14, that the parts of the curve of shade $(fac, f'a'c')$ and $(gbd, g'b'd')$, which are on the convex side of the surface, cast shadows upon it. These shadows begin at the points (c, c') and (d, d').

To find these shadows, we will, in the first place, find the shadow which the entire curve of shade would cast on the horizontal plane, under the supposition that the surface offers no obstruction to the light. This is done by finding where rays of light, drawn through the several points of the curve of shade, pierce the horizontal plane.

The upper part of the curve, which is on the convex side of the surface, casts the shadow $f''h''g''$—the lower part of the curve, on the convex side of the surface, casts the shadow $c''l''d''$—the parts of the curve on the concave side of the surface, cast the shadows $c''t''f''$ and $d''x''g''$. Assume now any horizontal circle below the points (c, c'), (d, d'); the one, for example, whose vertical projection is $k'mn$, and whose horizontal projection is the circle described with the centre A and radius Ak equal to mk'. The centre of this circle casts a shadow on the horizontal plane at m'—the circumference described with m' as a centre, and radius $m'p$, equal to mk', will be the shadow cast on the horizontal plane by the circumference of the assumed circle.

If through the points p and q, in which this shadow intersects the shadow cast by the curve of shade, rays of light be drawn, they will intersect, in space, the circumference of the horizontal circle and the curve of shade. The points in which these rays intersect the circumference of the horizontal circle, are points of shadow on the surface which are cast by the points in which the rays intersect the curve of shade.

Therefore, draw the horizontal projections of rays through the points p and q, and the points p' and q', in which they intersect the circumference $kp'q'$, described with the centre A and radius Ak, are the horizontal projections of two points of the shadow on the surface. These points are vertically projected at p'' and q''. By similar constructions, we may find any number of points in the shadow which the curve of shade casts on the surface.

If we take the lower horizontal circle FG, we shall find the points (t, t') and (x, x') where the shadows on the surface terminate. The curve $(cp't, c'p''t')$ and $(dq'x, d'q''x')$ may now be described. The curve $(cp't, c'p''t')$ intersects the meridian plane HA at the point (s, s').

We have thus found the lines on the exterior of this surface, which separate the dark from the illuminated part. They are the curve of shadow until it intersects the curve of shade, then the curve of shade until it returns to the interior surface, and then the curve of shadow cast on the surface by the curve of shade. The light does not fall on that portion of the surface which is above and within the parts of these curves.

Let us now find the shadow which the surface casts on the horizontal plane.

The elements of shade (I, I′) and (K, K′) cast lines of shadow on the horizontal plane, which are found by drawing rays of light through their upper extremities. Then find the shadow cast by the circumference of the circle whose vertical projection is FG. This shadow intersects the shadow cast by the curve of shade in the points t'' and x''. These points of shadow are cast by the points (t, t') and (x, x').

Find next, the shadow cast by the circumference of the upper circle of the surface. The centre of this circle casts a shadow at B′. With B′ as a centre, and a radius equal to BD, let the circumference of a circle be described—this circumference is the shadow sought.

If through the points y and z, in which this shadow intersects the shadow cast by the curve of shade, rays of light be drawn, they will intersect, in space, both the curve of shade and the upper circle of the surface. These points of shadow are therefore cast by the points (a, a') and (b, b') in which the curves of shade and shadow intersect upon the surface. The shadows $t''y$ and $x''z$ are cast by parts of the curve of shade.

The elements of shade (P, P′), and (L, L′) of the upper cylinder, cast shadows which are tangent to the circumference described with the centre B′. The circumference of the upper circle of the cylinder casts a shadow on the horizontal plane, which is also tangent to the shadows cast by the elements of shade.

OF THE SHADES AND SHADOWS OF THE ROMAN DORIC COLUMN.

44. This column is composed of three principal parts:—1st. The base; 2d. the shaft; and 3d. the capital.

That the figure may not be too complicated, we shall first find the shades and shadows on the base and shaft of the column, and then use a separate figure to determine those of the capital.

45. The base of the column (Pl. 10. Fig. 1) is composed of seven parts. 1. A rectangular prism, called a plinth, whose horizontal sections are squares: 2d. A solid of revolution, convex outward, called the lower torus: 3d. A cylindrical fillet: 4th. A solid of revolution, concave outward, called a scotia: 5th. A cylindrical fillet: 6th. The upper torus: 7th. A cylindrical fillet.

The shaft of the column rises from the upper fillet. For a short distance it is concave outward, then it becomes nearly cylindrical, and continues so to near its upper extremity, where it again becomes concave outward. The drawing is made on the supposition that the shaft is cylindrical between the parts of it which are concave outward.

46. The capital is composed of two distinct parts—the one a member whose horizontal sections are squares, having their centres in the axis of the column, the other a solid of revolution. The projections of the capital are shown in Fig. 2. The part of the capital whose vertical projection is $ww'o's's$, is the portion whose horizontal sections are squares—this part is called the abacus. The part of the abacus between rr' and nn' is called the cyma-reversa, or talon.

The part of the capital whose vertical projection is $ll'm'm$ is called the echinus. The part whose vertical projection is $kk'l'l$ is called the upper fillet. The part whose vertical projection is $ii'k'k$ is called the cavetto. The part whose vertical projection is $gg'h'h$ is called the neck. The part ff' is called the astragal, or colarino. The part whose projection is $dd'e'e$ is called the lower fillet. The entire column below the abacus and above the plinth is a solid of revolution, and the vertical projections, in the two figures, are meridian sections of its surface.

PROBLEM XVI.

To find the shades and shadows on the shaft and base of the Roman Doric column.

47. A semi-column will illustrate all the cases. Let the semicircle AMB (Pl. 10. Fig. 1) be the horizontal projection of the cylindrical part of the column, CND of the upper and middle fillets, EOF of the upper torus, GPH of the lower fillet, LQS of the lower torus, and the rectangle LR of the plinth.

The space included between the semicircle GPH, the horizontal projection of the lower fillet, and a semicircle described at equal distances from AMB and CND, limits the horizontal projection of the scotia. Let the vertical projections be made as in the figure.

Through the axis of the column, let a meridian plane of rays be drawn—VI is its horizontal trace. Through I draw I*a* at right angles to IV. The plane of rays, tangent to the cylindrical part of the shaft, touches it in an element, which pierces the horizontal plane at *a*; this element, which is the element of shade, can therefore be drawn.

The element of shade casts a shadow on the concave part of the shaft, beginning at the point (a, a'). This shadow is in the vertical plane of rays passing through the element of shade, and is therefore horizontally projected in the right line drawn through *a* perpendicular to I*a*. The vertical projection of this shadow is found by intersecting the shaft below a' by horizontal planes. Each plane will intersect the shaft in a circle, which being projected on the horizontal plane, the circumference will intersect the tangent through a. Then, projecting the point of intersection into the vertical plane, we determine a point of the vertical projection of the shadow.

The plane of rays tangent to the upper fillet, touches it in the element which is projected on the horizontal plane at b. This element of shade, and the circumference of the upper circle of the fillet, cast shadows on the upper torus. The shadow of the element is found by a construction similar to that used in finding the shadow on the foot of the shaft. Points of the shadow cast by the circle are found by first finding the shadow of the circle on a horizontal plane, and then finding the shadow which a horizontal section of the torus would cast on the same plane, and drawing rays through their points of intersection. The shadow cast by the circumference of the upper circle of the fillet on the torus, is made full, until it intersects at (h, h') the curve of shade, determined as in Prob. 12.

Passing the curve of shade on the upper torus, the construction for which is given in Prob. 12, we come next to the shadow which this curve casts upon the middle fillet.

To find this shadow, we intersect the torus and the fillet by a plane of rays $c'p'$, perpendicular to the vertical plane of projection. Through the point c', in which the trace of this plane intersects the vertical projection of the curve of shade, let a horizontal plane be passed—this plane will intersect the torus in a horizontal circle, and let this circle be projected on the horizontal plane. The horizontal projection of c' is at c, and (c, c') is a

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point of the curve of shade. The ray of light through this point pierces the fillet in the point (p, p'), which is a point of the curve of shadow. This shadow passes down upon the fillet obliquely, until it intersects the lower circle of the fillet or upper circle of the scotia, at (X, X'). That part of the upper circle of the scotia, from h'' to (X, X'), on which the light falls, casts a shadow on the scotia, which is found as in Prob. 13.

The ray of light passing through the point (X, X'), in which the shadow on the fillet intersects the upper circle of the scotia, determines the first point of shadow which the curve of shade on the torus casts on the scotia.

To find points of this shadow, intersect the torus and scotia by a plane of rays $e'f'$ perpendicular to the vertical plane; find the point (e, e') in which this plane intersects the curve of shade, and construct the curve in which it intersects the scotia. Through e draw the horizontal projection of a ray of light—the point f, in which it intersects the curve of the scotia, is the horizontal, and f' is the vertical projection of a point of shadow.

The shadow cast on the scotia by the curve of shade on the torus, terminates in the upper circle of the lower fillet. A part of this shadow, beginning at q, is seen in horizontal projection.

The element of shade on the lower fillet is (g, g'); and this element casts a shadow on the lower torus, beginning at its foot. Then, that part of the upper circle of the fillet, intercepted between the element of shade and the point in which it is met by the shadow on the scotia, casts a shadow on the lower torus. Then the shadow on the lower torus is cast by the curve of shade on the upper torus; and this shadow continues until it intersects the curve of shade determined as in Prob. 12.

PROBLEM XVII.

To find the shades and shadows on the capital and shaft of the Roman Doric column.

48. The part of the abacus, whose vertical projection is $s\,r'$, casts a shadow on the cyma-reversa.

The semi-reversa is composed of parts of cylinders whose elements are respectively parallel and perpendicular to the vertical plane of projection. Two of the cylinders intersect each other in a curve, whose horizontal projection is $q''n''$, and vertical projection qpn.

The line $(r''r''', rr')$ of the abacus casts a line of shadow on the cyma-reversa parallel to itself. Through (r'', r) draw a ray of light, and find the point in which it pierces the surface of the cylinder whose elements are parallel to the vertical plane: qq' is the horizontal line drawn through the point, and is the line of shadow required. The light falls on a part of a cyma-reversa below this line. Let a tangent plane of rays be drawn to the part of the cyma-reversa, whose elements are parallel to the vertical plane. It touches the cyma-reversa in the element pp', which is therefore an element of shade. This element of shade casts a shadow on the part of the cyma-reversa below it. This shadow is determined by finding the intersection of the tangent plane of rays with the cyma-reversa. It is the horizontal line near nn'.

The lower element $n\,n'$ of the cyma-reversa casts a shadow $o\,o'$ parallel to itself, on the part of the abacus below it.

We come next to the echinus, which is the half of a torus.

First, find the curve of shade on the echinus as in Prob. 12. Then, through the horizontal line of the abacus whose vertical projection is w draw a plane of rays: wc' is its vertical trace. Since this plane is perpendicular to the vertical plane of projection, the shadow cast by the line of the abacus is vertically projected in the trace wc'. The ray of light through the point (w'', w) pierces the neck of the column at the point (c, c').

Through the lower line of the abacus $(w''n''', w\,w')$ let a plane of rays be drawn. This plane intersects the echinus in a curve which is the shadow cast by the line $(w''n''', ww')$ on the echinus; this curve of shadow meets the curve of shade in the points x and y; leaving a part of the echinus in the light. The curve of shadow is found by intersecting the echinus by horizontal planes —each plane will intersect the echinus in a horizontal circle and the plane of rays in a right line: the point in which the right line intersects the circle is a point of the curve of shadow

The plane of rays passed through the line $(w''n''', w\,w')$ intersects the neck of the column in the curve $c't$, the cavetto in the curve u, and the fillet directly above, in the curve v.

The curve of shade on the echinus casts a shadow on the fillet below, which begins at k, and intersects at v the shadow cast by the lower line $(w''n''', ww')$ of the abacus.

The lower circle $k\,k'$ of the fillet, casts a shadow on the echinus, which begins at a, and intersects the shadows cast by the abacus in two points, one near a, the other at u.

From u to z the shadow is cast by the small part of the arc of the circle kk', which is in the light near v. The ray through z passes through the last point of the circle kk' which is in the light, and therefore passes through a point of the curve of shade on the echinus. From z to the line $i i'$, the shadow is cast by the curve of shade on the echinus.

The lower circle $i i'$ of the cavetto, casts a shadow on the neck of the column which intersects at t the shadow cast by the lower line $(w''n''', w w')$ of the abacus, and at t' the shadow cast by the curve of shade on the echinus; from t' to t'' the shadow is cast by the curve of shade on the echinus.

We come next to the astragal $f f'$. Find the curve of shade as in Prob. 12.

The curve of shade on the echinus casts a shadow on the astragal which intersects the curve of shade near g'. The curve of shade on the astragal casts a shadow on the fillet $dd'e'e$.

The lower circle $e e'$ of the fillet, casts a shadow on the shaft of the column, which intersects the element of shade at b.

The shadow cast by a curve of shade on a surface of revolution, may be found by drawing rays through its different points, and finding where they pierce the surface; but it is generally better to pursue the method adopted in Prob. 14.

OF THE HELICOID.*

49. The helicoid is a surface generated by a right line moving uniformly in the direction of another right line which it intersects, and having at the same time a uniform angular motion around it.

The fixed line is called the axis of the helicoid, and the moving line the generatrix.

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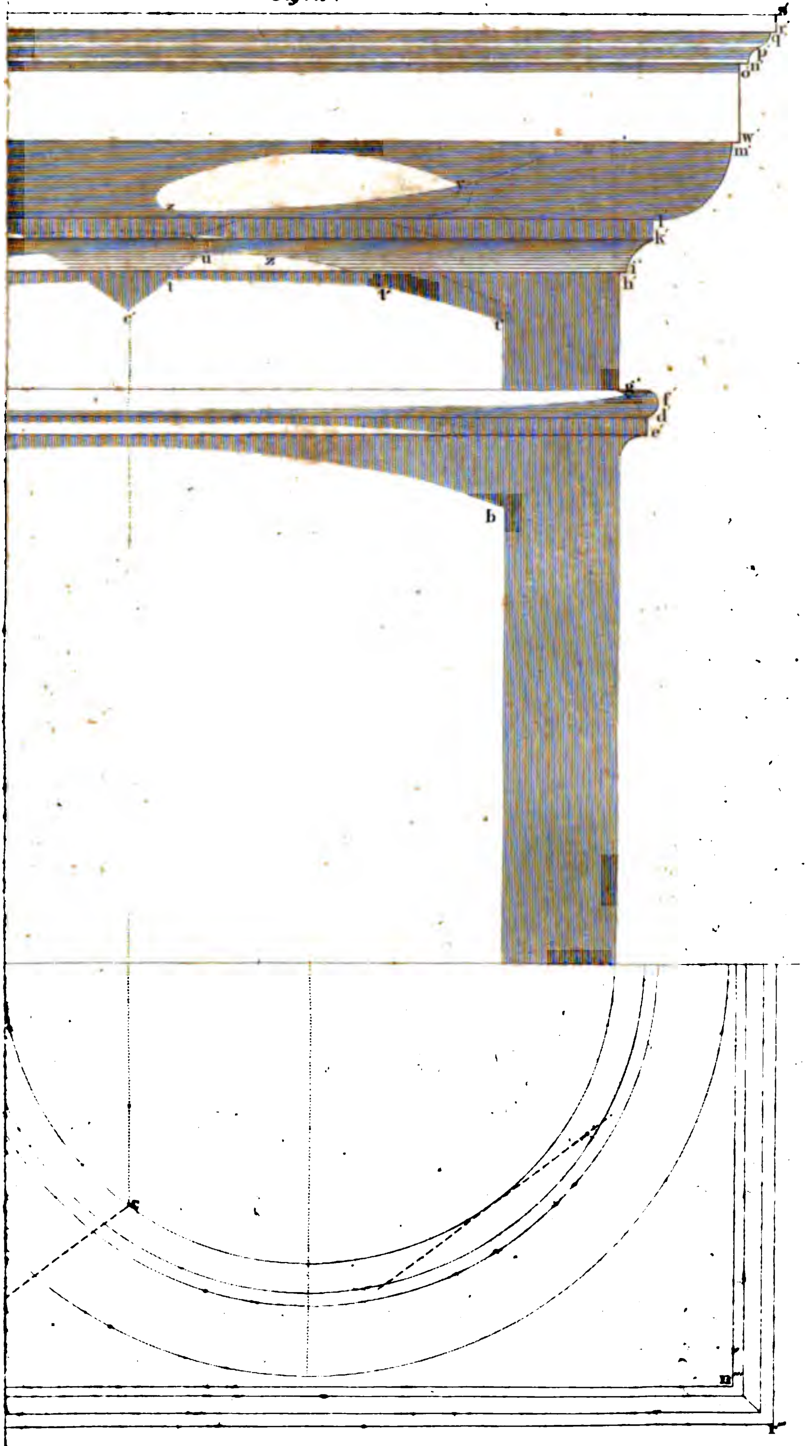
* The properties of the helicoid are used in the next problem, and although their discussion belongs rather to warped surfaces, than to a treatise on shades and shadows, yet it was thought best to give the properties here, as the student may not meet with them elsewhere.

Fig. 2.



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Let (A, BC) (Pl. 11, Fig. *n*) be the axis of a helicoid, and (DA, D′E) its generatrix.

By the definition of the surface, the generatrix has two motions; one in the direction of the axis (A, BC), the other around this axis, both of which are uniform.

Suppose the generatrix to move around the axis until it is horizontally projected in AF, and that during this revolution, it has moved in the direction of the axis a distance equal to EG.

Through F draw FPF′ perpendicular to the ground line, and make PF′ equal to EG—then F′G will be the vertical projection of the generatrix, from its new position; for every point of the generatrix has moved an equal distance in the direction of the axis.

By considering the circumstances of the motion, it is plain:

1st. That every point of the generatrix, except the one in which it intersects the axis, describes a curve—the curve so described is called a *helix.*

2d. That since the generatrix preserves the same position with the axis, all the points of the same helix will be equally distant from the axis, and consequently, the projection of any helix on a plane perpendicular to the axis, is in the circumference of a circle described about the centre A. The helix described by the point (D, D′) is horizontally projected in the semicircumference DIF.

When the generatrix has moved from one position to another, the motion in the direction of the axis is proportional to the part of the axis which the point of intersection has passed over; and all the points of the generatrix have moved equally in the same direction. The motion round the axis, or the angular motion of the generatrix, is proportional to the angle included between two planes passing through the axis and the two positions of the generatrix. The angle between these planes is measured by the arc of any circle described about the centre A. It may therefore be measured on the arc DIFR, or on the projection of any helix of the surface.

Since the motion in the direction of the axis, and the angular motion are both uniform, their measures will be proportional to each other: that is, when the point E of the generatrix has passed over a part of CB, the angular motion of the generatrix is measured by a like part of DIF.

Let the revolution of the generatrix (AF, GF') be continued until the point (F, F') shall be horizontally projected at D. The point G will then have moved to C, and the point (F, F') to (D, D''): GC and KD'' being each equal to EG. The revolution of the generatrix may be continued at pleasure.

The horizontal projection of the helix described by the point (D, D') is the circumference DIFR. It is required to find its vertical projection.

Let the semicircumference DIF be divided into any number of equal parts, say eight; and divide the distance $D'K$ into the same number of equal parts, and draw the parallel lines as in the figure.

Now, when the point (D, D') is horizontally projected at a, it will have ascended one-eighth of the distance $D'K$, and is, therefore, vertically projected at a'. Hence a' is a point in the vertical projection of the helix. Each of the other points of division also gives a point, and the vertical projection of the helix, from D' to F', can be drawn; the part $F'D''$ is easily found, since it has the same position with the horizontal plane KF', as the part $D'a'F'$ has with the plane $D'P$. The part

D″X has also the same position with the horizontal plane D″E, as the part D′aF′ has with the plane D′P.

The curve DQN, in which the horizontal plane of projection intersects the surface, is determined by finding the points in which the elements of the surface pierce the plane. This curve is the spiral of Archimedes—it extends indefinitely, unless we suppose the revolutions of the generatrix to be limited.

If through the point (D, D′), the line (DA, D′T) be drawn, making an angle with the axis of the surface equal to the angle D′EB, we may regard this line as the generatrix of a helicoid, similar in every respect to the one already described, excepting that it has a different position with the horizontal plane. We shall, for the sake of distinction, call the first the *upper helicoid*, and the second the *lower helicoid*. The helix described by the point (D, D′) is common to the two surfaces. Regarding the generatrices as indefinite, the surfaces are indefinite helicoids.

The helicoid is a warped surface, since the consecutive positions of the generatrix are not in the same plane (Des. Geom. 72).

It is now required to draw a tangent plane to the lower helicoid, at a point of the common helix.

50. Let (D, D′) be the point at which the tangent plane is to be drawn; and let us find its trace on the horizontal plane S′F′, drawn at a distance above the point of contact (D, D′), equal to the ascent of the generatrix in half a revolution.

The tangent plane must contain the element (DA, D′T), (Des. Geom. 89); and therefore, (S, S′) where it pierces the plane S′F′ is one point of the trace. But the tangent plane must also contain the tangent line to the helix at the point (D, D′) (Des. Geom. 88); and these two lines determine its position.

The helix (D*a*F, D′*a*F″) is on the surface of the right cylinder which projects it on the horizontal plane. Conceive a tangent plane to be drawn to this cylinder along the element which pierces the horizontal plane at D. The tangent line to the helix at (D, D′) will be contained in this plane.

Let the surface of the right cylinder be developed on the tangent plane. The semicircumference DIF will develop into a right line (Des. Geom. 131); and since the arcs D*a*, D*b*, &c, are proportional to the distances of the corresponding points of the helix above the horizontal plane, it follows that the helix will develop into a right line.

Then, lay off DV equal to the semicircumference DIF, and draw the perpendicular VF‴, and make it equal to PF″: we have then DF‴ for the development of the helix. But the tangent line at (D, D′) and the helix make the same angle with the horizontal plane; therefore, the helix developed coincides with the tangent line at (D, D′). Hence, the tangent line to the helix at the point (D, D′), pierces the horizontal plane S′F″ at the point (V, K), a distance from the plane DA equal to the semicircumference DIF. Therefore, SV is the horizontal projection of the trace of the tangent plane on the plane S′F″.

51. It is evident, from the similarity of triangles, that if the horizontal plane on which the trace is found, be taken above or below S′F″, the distance D′K will be to the semicircumference DIF, as the distance from D′ to the plane, is to the distance from the vertical plane DAF to where the tangent pierces the horizontal plane. Therefore, if the horizontal plane S′F″ were passed through D″,

making the distance from D′ equal to the ascent of the generatrix during an entire revolution, the distance DV would be equal to the entire circumference DIFR.

52. By considering the generation of the helicoid, it is evident that each helix makes a different angle with the horizontal plane; the limits of these angles being 90° and 0.

For, the helix described by the point in which the generatrix intersects the axis, is the axis itself, and is therefore perpendicular to the horizontal plane. The helix described by the point at an infinite distance from the axis may be considered horizontal, since the distance which the point moves in the direction of the axis is inconsiderable in comparison with its angular motion.

Since every tangent plane to the helicoid passes through an element (Des. Geom. 89), the limits of the angles which tangent planes make with the horizontal plane are 90°, and the angle made by the generatrix.

PROBLEM XIX.

Having given the projections of a screw and the direction of the light, it is required to find the shades and shadows on the different parts of the screw, and the shadow cast on a horizontal plane.

53. Let there be a vertical cylinder whose axis is (A, A′B), (Pl. 11) and the radius of whose base is AC.

Let the base of an isosceles triangle, whose vertical projection is C′D′E, be placed to coincide with the element of the cylinder which pierces the horizontal plane at C; the vertex D′ of the triangle being in the horizontal plane at D. The plane of the triangle will pass through the axis of the cylinder, and therefore the two equal sides will both intersect it.

Let the triangle be now revolved about the cylinder, having, at the same time, a uniform motion in the direction of the elements—the plane of the triangle continuing to pass through the axis.

When the triangle has been revolved half round the cylinder, suppose it to have ascended a distance equal to half its base. The vertex (D, D′) will then have the position (G,G′); G″G′ being equal to half the base. After the remaining half of the revolution is completed, the vertex of the triangle will have ascended a distance equal to its base, and will be vertically projected at F; FEH will then be the vertical projection of the generating triangle.

The cylinder about which the triangle has been revolved, is called the *cylinder of the screw.*

The solid generated by the triangle, in an entire revolution, is called *the thread of the screw.* The surface generated by the upper equal side of the triangle is called the upper surface of the thread; and the surface generated by the other equal side is called the lower surface of the thread. These surfaces are portions of helicoids (49). The helicoids may be considered indefinite, by supposing the equal sides of the triangle to be indefinitely produced.

The helix described by the vertex (D, D′), is called the outer helix of the screw; and the one described by either extremity of the base, is called the inner helix. The curves described by the intermediate points are called helices.

The vertical projections of the outer and inner helices are constructed as in Art. 49. The parts which lie in front of the plane DAG are made full, those which lie behind it are dotted. The horizontal projection of the outer helix is the circumference described with the radius AD, and of the inner helix, the circumference described with the radius AC.

The outer helix rises from the horizontal plane at D, and passes behind the plane DAG at (G, G′). The inner helix rises above the horizontal plane at I, and passes in front of the plane DAG at (C, E).

If the triangle EFH be again revolved around the cylinder of the screw, it will generate a second thread; and every new revolution will give an additional thread.

The base of the triangle, or the distance which it has ascended in an entire revolution, is called the *distance between the threads*.

The curve into which the surface of the screw intersects the horizontal plane is found as in Art. 49, excepting that the elements are differently inclined to the horizontal plane, which, however does not vary the principles of the construction.—It is the curve D*z*I.

The screw is usually worked in to a block of wood called a fillet. We have taken the fillet octagonal; the octagon 4 1 2 3 5 is its horizontal projection, and the rectangle above the threads its vertical projection.

It is now required to find the curve of shade on the lower surfaces of the threads.

Let a plane be drawn tangent to the lower surface of the thread, at the point (D, F) (50), and let its trace be constructed on the horizontal plane L′H, taken at a distance above the point of contact equal to the half distance of the threads.

The point (L, L′), in which the element passing through the point of contact pierces the horizontal plane L′H, is one point of the trace of the tangent plane. Drawing through D the perpendicular DO, and making it equal to the semicircumference D*a*G, we have the distance from the plane DAG, at which the line tangent to the helix at (D, F) pierces the horizontal plane L′H (50). This distance should have been laid off in front of the plane DAG, but this could not be done, for want of room on the paper. If however, we suppose a line drawn through L and the point O, taken at a distance equal to DO in front of the plane DAG, it will make an angle with LD equal to the angle DLO. Hence, if *k*L*k*′ be drawn, making the angle DL*k*′ equal to the angle DLO, it will be the horizontal projection of the trace of the tangent plane. The tangent plane cuts the axis of the screw at (A, I′).

Now, if this plane be a plane of rays, the point of contact (D,F) will be a point of the curve of shade.

To ascertain if it be a plane of rays draw through (A, I′) a ray of light; its projections are AK and I′K′.

If the tangent plane be a plane of rays, the ray of light having one point in common, will coincide with it; and if it coincide with it, it will pierce the horizontal plane L′H in the trace *k*L*k*′. But it pierces this horizontal plane at (K, K′) out of the trace—therefore, the tangent plane is not a plane of rays.

Let the ray (AK, I′K′) be revolved about the axis of the screw; it will generate the surface of a right cone whose vertex is (A, I′); and the point (K, K′) will describe, in the horizontal plane K′L′H, the arc of a circle of which *k*K*k*′ is the horizontal projection. The tangent plane to the screw will intersect the surface of this cone in two elements whose horizontal projections are A*k* and A*k*′.

These lines may be regarded as the ray (AK, I′K′) after it has been revolved about the axis of the screw to coincide with the tangent plane.

Let the tangent plane be now revolved around the axis of the screw, in such a manner as to continue tangent along the outer helix, until the revolved ray A*k* shall be horizontally projected in AK. In this revolution of the tangent plane every point of the line A*k* will ascend equally; therefore, after the counter revolution the vertical projection of the line will be parallel to I′K′; hence, the line itself will be a ray of light.

In the revolution of the tangent plane, every point has an equal angular motion, and this motion is measured by the arc *o*D*b*; the point of contact (D, F) having the same angular motion, if we lay off D*ba* equal to *o*D*b*, or what is the same, *ba* equal to *o*D, we find *a*, the horizontal projection of the point of contact when the tangent plane is a plane of rays. Projecting *a* into the vertical projection of the outer helix, we have one point (*a*, *a*′) of the curve of shade.

If we revolve the tangent plane from D towards *o*, as we are at liberty to do, causing it to descend along the outer helix but continuing tangent to the surface, until the line A*k*′ shall be horizontally projected in AK, the plane will, for the reasons given above, become a plane of rays, and the point of contact, a point of the curve of shade. The angular motion, in this revolution, will be measured by *bc*—hence, laying off D*c*′ equal to *bc*, we have *c*′, the horizontal projection of the point of contact. Projecting the point *c*′ into the vertical projection of the outer helix at *c*″, determines (*c*′, *c*″), a point in a second curve of shade. Hence we see that two tangent planes of rays can be drawn touching the same thread of the screw in points of the outer helix.

Let a tangent plane be drawn to the under surface of the thread of the screw, at the point (C, E) of the inner helix; and let its trace be constructed on the horizontal plane K'L'H, at a distance above the point of contact equal to the ascent of the generating triangle in an entire revolution. The point (L, L'), in which the element through the point of contact pierces the horizontal plane L'H, is one point of the trace, and making CP equal to the circumference C*ef*I*f* (51), and considering P in front of the plane DAG, we have a second point of the trace, it being the point in which the tangent line to the helix at (C, E) pierces the plane L'H. Drawing LP, and then the line L*d*', making the angle DL*d*' equal to the angle DLP, we determine *d*'L*d* the trace, on the plane L'H, of the tangent plane to the inner helix at the point (C, E).

This tangent plane cuts the axis of the screw at the point (A, I'), and since the ray through this point does not pierce the horizontal plane K'H in the trace of the tangent plane, it follows, that the tangent plane is not a plane of rays. Let the ray (AK, I'K') be revolved about the axis of the screw as before—it will generate the surface of a right cone, which the tangent plane will intersect in two elements whose horizontal projections are A*d* and A*d*'.

The tangent plane is now revolved to become a plane of rays, as in the last case. The angular motion, when it ascends, is *e*C*g*, and when it descends, is *fg*. Therefore, making *ge*' equal to C*e*, and C*f*' equal to *gf*, we have *e*' and *f*' the horizontal projections of the two points of tangency. The point *e*' is vertically projected on the inner helix at *e*'', and the point *f*' at *f*''—hence, two points in each of the curves of shade are found. It is now required to find intermediate points.

Any element of the lower surface of the thread whose horizontal projection passes between the points *a* and *e*', will contain an intermediate point of the curve of shade.

Assume Ah for the horizontal projection of such an element.

Projecting the point h into the outer helix at h', and the point i into the inner helix at i', and we find $Ri'h'$ the vertical projection of the element.

Through (h, h'), the upper extremity of the element, draw the horizontal plane $M'h'$, and through the point (A, R) draw a ray of light (AM, RM'); this ray pierces the horizontal plane $M'h'$ at the point (M, M').

If through the element (Ah, Rh') we suppose a plane of rays to be passed, its trace on the plane $M'h'$ will be the line of which Mh is the horizontal projection; and since the surface of the screw is a warped surface, this plane will be tangent to it at some point of the element (Des. Geom. 229): the point of contact is a point of the curve of shade.

Let us suppose for a moment this point of contact to be found, and that it is (u, u'). At a distance above the point of contact equal to half the distance of the threads, let the horizontal plane $N'H'$ be drawn. The trace of the plane of rays on this plane is parallel to its trace on the plane $M'h'$; therefore, Nh'' drawn parallel to Mh is the trace on the plane $N'H'$.

If then, through the point u, a perpendicular be drawn to Au, and produced till it meets Nh'' in Q, the distance uQ will be the semicircumference of the circle whose radius is Au (50). But $h''u$ is the horizontal projection of a part of the element intercepted between two horizontal planes, at a distance from each other equal to the half distance of the threads; and since all the elements make the same angle with the horizontal plane, this projection is equal to the projection of $D'E$, that is, to DC or hi. Hence, if through i, iS be drawn perpendicular to Aih, we shall have two similar triangles, $h''uQ$ and hiS, having in each an equal homologous side $h''u$ and hi; hence the sides uQ and iS are equal. But the line uQ has been proved equal to the semicircumference of the circle passing through the point of contact; therefore, iS is also equal to this semicircumference.

After having drawn the trace Mh, of the plane of rays, we have only to draw iS perpendicular to Aih and find the radius of the circle of which iS is the semicircumference. This is easiest done by laying off iS from D, on DO, and drawing through the extremity of the line a parallel to the line joining A and O—the distance cut off from D, on DA, is the radius required; for the semicircumferences are to each other as their radii. Then, with this radius, and A as a centre, describe an arc; the point u, where it cuts Ah, is the horizontal projection of the point of contact, and this point being projected into the vertical projection of the element at u', the vertical projection of the point of contact is also determined. An intermediate point in the curve of shade on the other side of the plane DAG, may be found by a similar construction.

Since the light has the same position with the lower surfaces of all the threads, the curves of shade upon them will be directly over each other; therefore, if a vertical line be drawn through (a, a'), the points in which it cuts the outer helices will be points of the curves of shade, and the vertical line through (e', e''), will determine corresponding points of shade on the inner helices.

It is now required to find the shadow which any thread will cast on the surface of the thread directly below it. The curve of shade, and a part of the outer helix, will cast shadows on the thread below them. The shadow will begin at the point where the curve of shade meets the inner helix.

To find the shadow cast by the curve of shade, we first find the shadow which the curve of shade would cast on the horizontal plane if the screw were removed. This is done by drawing rays of light through its several points, and finding where they pierce the horizontal plane. We then take any element of the upper surface, near to (e', e''), on which we suppose a shadow will fall, and find the shadow which it would cast on the horizontal plane. Through the point p, where these shadows intersect, if we suppose a ray of light to be drawn, it will intersect both the element and the curve of shade. The point where it meets the element is the point of shadow on the element, and the point where it meets the curve of shade is the point casting the shadow. The ray through p gives (p', p'') for the point of shadow.

To find the shadow cast by the helix. Take any element on which we suppose the shadow will fall, and through it pass a plane of rays. Find the point in which this plane cuts the outer helix, directly above, and through this point draw a ray of light, where it intersects the element is a point of the required shadow.

Let Aq be the horizontal projection of an element on which it is supposed the shadow will fall, and q' v its vertical projection. Through (q, q'), the foot of the element, let the horizontal plane q'T' be drawn. Through (A, v), the point in which the element intersects the axis, let a ray of light be drawn; this ray pierces the horizontal plane through (q, q') at (T, T')—therefore qT is the trace, on the plane q'T', of the plane of rays through the element. It is now required to find the point in which this plane of rays cuts the outer helix (D$aq$G, FX.)

Through the axis of the screw let a plane be passed perpendicular to the plane of rays—its horizontal trace is Ar perpendicular to qT. Let the plane of rays and the plane whose horizontal trace is Ar be revolved about the axis of the screw until Ar becomes parallel to the ground line; and suppose the plane of rays to have at the same time an ascending motion in the direction of the axis, such that (Aq, vq') shall continue an element of the screw. When Ar is parallel to the ground line, the plane of rays will be perpendicular to the vertical plane of projection, and the angular motion will be measured by the arc rG.

From q laying off qq'' equal to rG: Aq'' will be the horizontal projection of the element after the revolution, and $q'''s'$ will be its vertical projection. But, since the element is a line of the plane of rays, which, in its revolved position is perpendicular to the vertical plane; the projection of the element is the vertical trace of the plane (Des. Geom. 24).

Therefore s', the point in which the line $q'''s'$ intersects FX, is the vertical projection of the point in its revolved position, in which the plane of rays cuts the outer helix. The horizontal projection of this point is at s. In the counter revolution, the point (s, s') has the same angular motion as the other points of the element —therefore, laying off ss'' equal to rG, and projecting s'' into the outer helix at s''', determines (s'', s''') the point in which the plane of rays cuts the outer helix.

Through this point draw a ray of light—the point (n, n'), where it intersects the element, is a point of the curve of shadow. By similar constructions we may find other points of the curve.

If through any element of the upper surface of a thread, a plane of rays be drawn, and its trace constructed on the horizontal plane through the foot of the element, the horizontal projections of these traces will all pass through the point T.

For, suppose we take the element whose vertical projection is $q'''s'$. Through q''' draw the horizontal plane $q'''T''$. The ascent of the foot of this element in revolving from the position (Aq, vq') is equal to $T'T''$, and the point v has ascended the same distance. Therefore, the vertical projection of the ray drawn through the point in which the element intersects the axis, will intersect the vertical line $T'T''$ at T'', which is in the horizontal plane $q'''T''$; and as the same may be shown for all other elements, it follows, that the horizontal projections of all the traces will pass through the point T. Having found several points of the curve of shadow, let its two projections $e'p'nt$, $e''p''n't$ be drawn.

The shadows on the threads directly above, are similar to the one already found, and are all horizontally projected in the curve $e'p'n\,t$.

Drawing through t' the vertical projection of a ray of light, the point l, where it meets the outer helix above, is the vertical projection of the point which casts the shadow at (t, t'); and $a'l$ is the vertical projection of the part of the outer helix which casts a shadow on the thread.

There are similar shadows on the other side of the plane of rays KAT, which are constructed in the same manner as those already found.

To find the shadow cast on the screw by the fillet:

The lower lines of the fillet which are towards the source of light and whose horizontal projections are 12, 23, and 14, will cast shadow on the screw.

Let it be required to find the shadow cast by the line whose horizontal projection is 12.

Through this line suppose a plane of rays to be passed —the curve in which it intersects the surface of the screw is the curve of shadow required. The plane of rays through the axis of the screw, cuts the lower line 12 of the fillet at the point (y, y'). Through y' draw the vertical projection of a ray—the point y'', where it intersects the axis of the screw, is the point in which the plane through 12 cuts the axis.

Every plane passing through the axis of the screw will intersect the surface in an element, and the plane of rays in a right line passing through the point (A, y''); the point in which these lines intersect is a point of the shadow.

The plane whose trace is KA intersects the thread on which the shadow falls in the element (bg, $b'g'$), and the plane of rays in the line whose vertical projection $y'y''$; hence their point of intersection is a point of the curve of shadow. The plane LA gives the point of shadow on the element parallel to the vertical plane.

The plane of rays through 23, cuts the axis of the screw at X″. Having found a sufficient number of points, let the vertical projection of the curve be drawn. The shadow passes off the thread at m. The horizontal projection is easily found, but is not made, lest it should render the figure on the horizontal plane too complicated.

To find the shadow cast by the screw on the horizontal plane:

This shadow lies on both sides of the plane KA; but as the construction for the two parts is the same, it will only be necessary to find one of them.

The shadow on the horizontal plane begins at the point z, where the curve of shade on the thread intersects the horizontal plane. The curve of shade between this point and the outer helix, casts a shadow on the horizontal plane; this shadow is found by drawing rays through the points of shade, and finding where they pierce the horizontal plane.

The part of the outer helix, between the points a'' and t, next casts a shadow. This shadow is found by drawing rays through the points of the helix, and finding where they pierce the horizontal plane. The point (t, t') casts its shadow at t'', and $z\,t''$ is the shadow cast by the curve of shade and outer helix.

The ray through (t, t') intersects the outer helix of the next higher thread at l. The point whose vertical projection is l casts a shadow on the screw at (t, t'), and on the horizontal plane at t''. The part of the helix from l to the point where it intersects the shadow on the thread, will cast a shadow on the horizontal plane, which is found by drawing rays through the points of the helix.

We then pass to the helix of the next higher thread, and so on until we arrive at the thread on which the shadow of the fillet falls.

Having before found m, the vertical projection of the point in which the shadow of the fillet intersects the outer helix, we find the shadow of this point on the horizontal plane which is at m'. At this point the shadow of the fillet on the horizontal plane begins. This shadow is found too easily to require particular explanation.

54. The brilliant point of the surface of the helicoid, is found by bisecting the angle between a ray of light and a line drawn to the eye, and then drawing a plane perpendicular to the bisecting line and tangent to the surface. The details of the construction are left as an exercise for the student.

55. In all the constructions which have been made, the rays of light have been supposed parallel. It may be well to consider how the constructions would have been made had the rays been divergent or convergent.

If we suppose the rays of light to emanate from a luminous point at a finite distance, and to fall upon an opaque body, the rays will be divergent.

Let us suppose a cone to be drawn tangent to the opaque body, of which the luminous point shall be the vertex. It is plain,

1°. That the line of contact of this cone with the opaque body will be the line of shade.

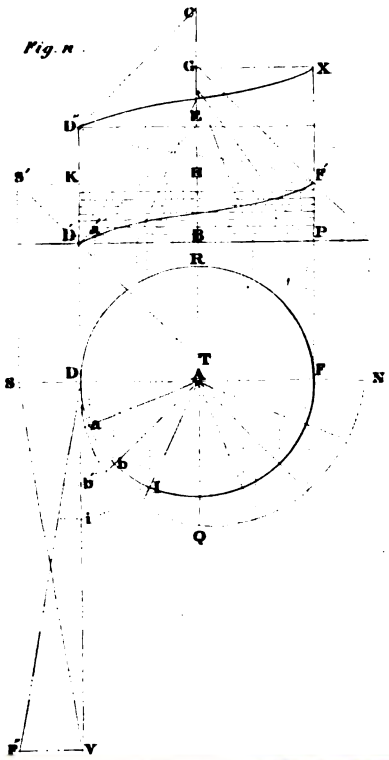
2°. That the part of space included within the surface of this cone, and lying on that side of the opaque body opposite to the source of light, will be the indefinite shadow of the opaque body.

3°. That the shadow of the opaque body on any surface, will be the intersection of that surface with the tangent cone of rays.

56. If we suppose the light to emanate from a luminous body, we may suppose a cone drawn tangent to the luminous body, and to any opaque body whose shade and shadow are to be determined. The curve of contact on the opaque body will be the line of shade, and the part of space within the surface of the tangent cone and on the side of the opaque body opposite the source of light, will be the indefinite shadow. When the opaque body is the largest, the luminous body and the vertex of the tangent cone will be on the same side of the opaque body, and the rays of light will be divergent; but when it is the smallest, the opaque body will be between the luminous body and the vertex of the cone, and the rays of light may be considered as converging to the vertex of the tangent cone.

In all the cases in which the rays of light are not parallel, the problems in Shades and Shadows will be solved by finding the contact of a cone with an opaque body, and the intersection of this cone with any surface on which the shadow falls.

Fig. 11.



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LINEAR PERSPECTIVE.

CHAPTER I.

57. HAD we no knowledge of objects other than what is derived through the medium of sight, we should suppose them to differ from each other in two respects only —form and colour. Objects having different forms, or different colours, produce different effects upon the eye; but objects of the same form and colour cannot be distinguished from each other without the aid of the other senses.

58. When we view an object, all the points of it which are seen are supposed either to emit or reflect rays of light which fall upon the eye; and it is through the medium of these rays that we derive the idea both of its form and colour.

59. Perspective is the art of representing objects on a surface, in such a manner that the representations shall present to the eye, situated at a particular point, the same appearance as is presented by the objects themselves. The representation of an object so made is called its *perspective.*

60. Let us now suppose that we are viewing an object in space, and that a transparent plane is placed between us and the object.

Every point of the object which is seen, is supposed to emit a ray of light that falls upon the eye, and each ray pierces the transparent plane in a point. If to each point so determined, a proper colouring be given, the representation or picture, on the transparent plane, will present to the eye the same appearance as the object itself. Such representation is, therefore, the perspective of the object.

61. The rays of light coming from the different points of the object to the eye, are called *visual rays;* and the plane on which the representation is made, is called the *Perspective Plane.*

62. The art of Perspective is therefore divided into two parts.

1°. To find the points in which the visual rays pierce the perspective plane, which determines the general outline of the perspective.

2°. So to shade and colour this outline, that it shall appear in every respect like the object itself.

The first part is called *Linear Perspective.* It embraces that portion of perspective that is strictly mathematical, and which will form the subject of the following treatise.

The second part is called *Aerial Perspective.* This branch of the art belongs to the draftsman and the painter, and is to be learned by a careful study of the objects of nature, under the guidance of an improved and cultivated taste.

63. As it is the end of perspective to represent objects as they appear in nature, such a position ought to be given to the perspective plane as will enable us to conceive, most easily, of the positions of objects from viewing their perspectives.

This we can do with the least difficulty, when the perspective plane is taken parallel to the principal lines of the object.

Of the objects in nature, the larger portion of lines are vertical; therefore, in most perspective drawings the perspective plane has a vertical position.

64. Before an object can be put in perspective three things must be given, or known.

1°. The place of the eye.

2°. The position of the perspective plane.

3°. The position of the object to be put in perspective.

65. By considering what has already been said, we may deduce the following principles:

1°. If through the eye and any point in space, a visual ray be drawn, the point at which it pierces the perspective plane is the perspective of the point in space.

2°. If through the eye and all the points of a right line, a system of visual rays be drawn, they will form a plane passing through the eye and the right line; this plane is called a visual plane, and its trace on the perspective plane is the perspective of the right line.

3°. If through the eye and all the points of a curve, a system of visual rays be drawn, they will, in general, form the surface of a cone, the vertex being at the eye; this cone is called a visual cone, and its intersection with the perspective plane is the perspective of the curve.

66. In determining the perspective of an object it is unnecessary, and indeed impracticable, to draw visual rays through all its points that are seen, and to construct the intersections of these rays with the perspective plane. We therefore select the prominent points and lines only, such as the vertex and edges of a pyramid, the vertex of a cone, the edges of a prism, &c.: and having put these lines in perspective, we have, in fact, determined the perspective of the body.

67. If through the eye a system of visual rays be drawn tangent to the object to be put in perspective, they will, in general, form the surface of a visual cone tangent to the object; the line of contact is called the *apparent contour of the object;* and the intersection of the surface of this cone with the perspective plane, is the boundary of the perspective of the object.

We shall now apply these principles in finding the perspectives of objects.

PROBLEM I.

Having given a cube and its shadow on the horizontal plane, it is required to find the perspective of the cube and the perspective of its shadow.

68. Let DEGF (Pl. 12) be the horizontal projection, and D'E'G'F' the vertical projection of the cube. And let D*ehg*G be the shadow cast on the horizontal plane.

Let (A, A') be the place of the eye, and BC, BC' the traces of the perspective plane.

The perspective of the cube and its shadow, after they shall have been found, will be projected on the planes of projection in the traces of the perspective plane (Des. Geom. 24); but in order to exhibit their perspective truly to the eye, it must be presented as it appears on the perspective plane.

For this purpose we remove the perspective plane parallel to itself, any convenient distance, as BB', and then revolve it about its vertical trace B'E' until it coincides with the vertical plane of projection. The place of the eye is supposed to be moved with the perspective plane, and to have the same relative position with it after it has been revolved.

Through the angle (D,D′) of the cube, draw the visual ray (AD, A′D′); this ray pierces the perspective plane at the point (d, a'), and after the plane has been moved and revolved, the perspective of the point is at d''.

Through (E, D′) draw the visual ray (AE, A′D′); this ray pierces the perspective plane at (a, a'), and determines a'' the perspective of (E, D′). Hence $d''a''$ is the perspective of DE.

The visual ray through the point (F, F′) pierces the perspective plane at the point (f', f''), and determines e, the perspective of the point (F,F′). The visual ray through the point (E,E′) pierces the perspective plane at (a, k), and determines k', the perspective of the point (E, E′). Hence $a''k'$ is the perspective of the edge (E, D′E′) of the cube.

The visual ray through (D, E′) pierces the perspective plane at (d, k), and determines p, the perspective of the point (D, E′). Hence pk' is the perspective of the edge (DE, E′) of the cube, $d''p$ of the edge (D, D′E′), and the square $a''k'pd''$ of the front face of the cube, which is parallel to the perspective plane.

Having determined, by similar constructions, the perspectives of the other angles of the cube, we see that the trapezoid $d''pq'f$ is the perspective of the face which is horizontally projected in the line DF; the trapezoid $pk'n''q'$ the perspective of the upper face of the cube; the trapezoid $d''a''n'f$ the perspective of the base of the cube; the square $fn'n''q'$ the perspective of the back face of the cube which is parallel to the perspective plane; and the trapezoid $a''k'n''n'$ the perspective of the face of the cube which is horizontally projected in the line EG.

There are but three faces of the cube which are seen, viz. the upper face, the front face parallel to the perspective plane, and the face which is horizontally projected in the line FD. The perspectives of the lines bounding these faces are made full in the perspective plane. The perspectives of the other edges are dotted, in order to show how the cube would appear if it were a transparent body.

The perspective of the shadow on the horizontal plane is found by finding the perspectives of the points e, h, and g, and drawing the lines $d''e$, eh, hg and gn'.

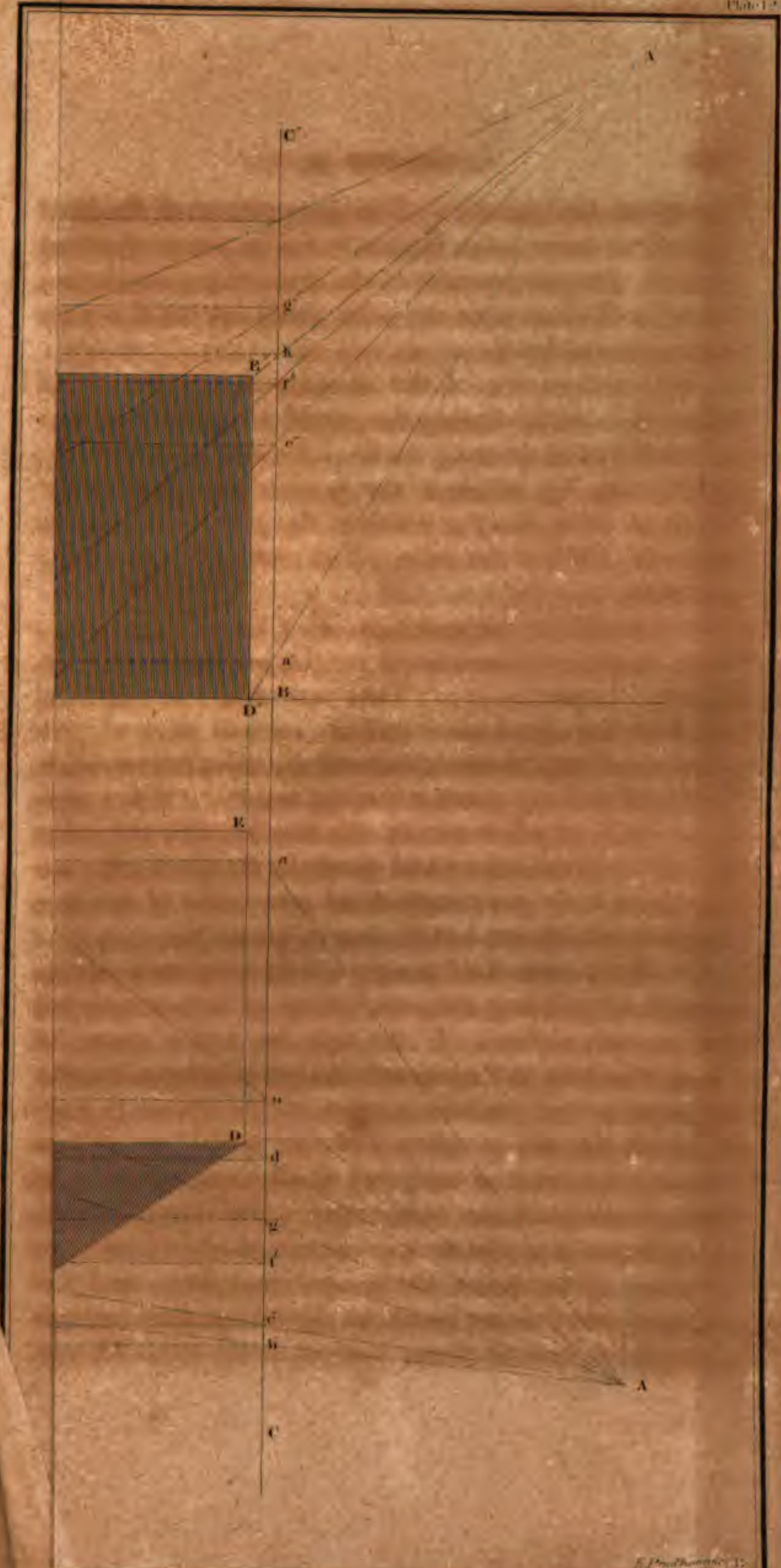
The line hg, which is the perspective of hg, intersects at c the line fq, which is the perspective of the edge (F, F′G′) of the cube. The part cg of the line hg not seen.

If through the vertical edge (F, F′G′) of the cube, a visual plane be passed, it will cut the line hg in the point whose perspective is c. That part of the line hg lying between the visual plane and the vertical plane of projection will not be seen, because the cube intervenes.

The visual ray passing through the point whose perspective is c, will intersect the line hg, and the edge (F, F′G′) of the cube; and generally *the visual ray passing through the point in which the perspectives of two lines intersect, will intersect both the lines in space.*

69. This method of perspective is often used advantageously in finding the perspectives of bodies bounded by curved surfaces. If through the eye a plane be passed tangent to the object to be put in perspective, the point of contact will be a point of the apparent contour of the object (67), and the visual ray drawn through the point of tangency will determine a point in the boundary of the perspective. For example, if it were required to find the perspective of an ellipsoid, the position of the body, the perspective plane, and the place of the eye being given, we should first pass through the eye a system of planes tangent to the ellipsoid, and

Plate 12



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then draw visual rays through the points of contact; the points in which these visual rays pierce the perspective plane are points in the boundary of the perspective of the ellipsoid.

CHAPTER II.

OF THE METHOD OF PERSPECTIVE BY MEANS OF DIAGONALS AND PERPENDICULARS.

70. The point from which the eye is supposed to view an object put in perspective, is called the *point of sight;* and the projection of this point on the perspective plane is called the *centre of the picture.*

71. If through the point of sight a right line be drawn parallel to any right line in space, the point in which it pierces the perspective plane is called *the vanishing point of that line.*

Hence, all parallel right lines have the same vanishing point: for a right line parallel to one of them will be parallel to all the others.

Hence also, all lines perpendicular to the perspective plane have their vanishing point at the centre of the picture.

Regarding a right line as indefinite in length, its vanishing point is a point of its perspective. For, the parallel through the point of sight is contained in the visual plane passing through the given line; therefore, the point at which it pierces the perspective plane is in the trace of the visual plane. But the trace of the visual plane is the perspective of the given line (65); therefore, the vanishing point of a line is a point of its perspective.

When a line is indefinite in length, its perspective is called the *indefinite perspective of the line.*

It follows from what has just been shown, that *the centre of the picture is common to the perspectives of all lines which are perpendicular to the perspective plane.*

72. The point in which a line pierces the perspective plane, is also a point of its perspective. When therefore we have found this point, and the vanishing point, the perspective of the line can be drawn.

If a system of lines be parallel to the perspective plane, the line drawn through the point of sight parallel to them, will also be parallel to the perspective plane; and hence, the vanishing point of the system will be at an infinite distance from the centre of the picture.

Now, since the perspectives of all the lines must pass through their vanishing point, and since their vanishing point is at an infinite distance from the centre of the picture, it follows, that the perspectives of lines so situated are parallel to each other. We can prove in another way that these perspectives are parallel.

For, the visual plane passing through either of the parallel lines will intersect the perspective plane in a line parallel to the system of lines; hence, the traces of the several visual planes will be parallel to each other. But these traces are the perspectives of the given lines; therefore, when a system of parallel lines is parallel to the perspective plane, their perspectives will be parallel to each other.

It follows, from what has been said, that when a line is parallel to the perspective plane, the perspective of the line will be parallel to the line itself.

73. If we have a system of parallel right lines, which are not parallel to the perspective plane, their perspectives will meet in a common point. For, the system of parallel lines will have a common vanishing point (71); and the indefinite perspective of each line will pass through this point.

The same fact may also be proved otherwise. For, suppose visual planes to be drawn through the lines. These visual planes will all intersect each other in a line passing through the eye and parallel to the system of lines; the point in which this parallel pierces the perspective plane will be common to the traces of all the visual planes, and consequently, to the perspectives of all the lines.

74. If in the perspective plane, a horizontal line be drawn through the centre of the picture, it will be the locus of the vanishing points of all horizontal lines. For, all horizontal lines drawn through the point of sight, are contained in the horizontal plane through the point of sight, and will therefore pierce the perspective plane in the trace of this horizontal plane, which trace is the horizontal line drawn through the centre of the picture.

75. Horizontal lines, which make angles of 45° with the perspective plane, are called *diagonals*. If through the point of sight two diagonal lines be drawn, one on each side of the perpendicular to the perspective plane, the points in which they pierce the perspective plane are called the vanishing points of diagonals. These points are in the horizontal line drawn through the centre of the picture, and at equal distances from the centre.

76. Since the line which determines either of these points makes an angle of 45° with the perpendicular through the point of sight, as well as with the perspec-

tive plane, it follows that the distance from the centre of the picture to the point of sight is equal to the distance from the centre of the picture to the vanishing point of diagonals.

Let AB (Pl. 13. Fig. *n*) be the trace of the perspective plane, which is perpendicular to the plane of the paper.

Let C be the horizontal projection of the point of sight, and C′ its projection on the perspective plane: C′ is then the centre of the picture (70). Through the point of sight conceive two diagonal lines to be drawn—their horizontal projections are C*a* and C*b*, which make angles of 45° with the perpendicular CD. But these diagonals pierce the perspective plane at the points *a*′ and *b*′ in the horizontal line *a*′C′*b*′, and these points, by definition, are the vanishing points of diagonals. Since the angles DC*a* and DC*b* are each 45°, D*a* and D*b* are each equal to CD. But C′*a*′ and C′*b*′ are each equal to D*a* or D*b*, and consequently to DC, which is the distance of the point of sight from the centre of the picture.

77. By considering what has been said, we have this general rule for determining whether the vanishing point of a diagonal is on the right or left of the centre of the picture. Through any point of the diagonal, in front of the perspective plane, draw a perpendicular to the perspective plane. Now, when the part of the diagonal intercepted between the point and the perspective plane lies on the right of the perpendicular, the vanishing point of the diagonal is on the right of the centre of the picture; but when it lies on the left of the perpendicular, the vanishing point is on the left of the centre of the picture. The rule is reversed when the point through which the perpendicular is drawn is behind the perspective plane.

78. The principles on which the perspectives of points are found by the method we are now explaining, are these,

1°. That the perspective of every point of a right line is found somewhere in the indefinite perspective of the line.

2°. That the perspectives of two points of a right line determine the perspective of the line. From the first principle it follows, that if two lines be drawn through any point in space, and their perspectives determined, the intersection of their perspectives will be the perspective of their point of intersection.

Let *c* (Pl. 13, Fig. *n*) be a point in the horizontal plane at a distance *c d* behind the perspective plane.

The perpendicular through *c* pierces the perspective plane at *d*; and since *C′* is the vanishing point of perpendiculars, *dC′* is the perspective of the perpendicular *cd*. The diagonal through *c* pierces the perspective plane at *f*, and the diagonal has its vanishing point at *a′*, therefore *fa′* is its perspective. But the perspective of the point *c* is found both in *dC′* and in *fa′*; it is therefore at *c′* their point of intersection.

We could have determined the perspective of the point *c* by finding the perspectives of any other two lines passing through it; but it is better to use the perpendicular and diagonal than other lines, because their perspectives are more readily found.

79. The perspective plane being placed between the eye and the object, the perspective of the object will lie above the ground line, and its horizontal projection will be behind the perspective plane. Now, when the horizontal plane is revolved in the usual way, to coincide with the perspective plane, the perspective, the horizon-

tal projection, and the vertical projection of the object will occupy the same part of the paper.

To avoid this inconvenience in the constructions, we revolve the horizontal plane of projection about its intersection with the perspective plane, in such a manner that the part behind the perspective plane shall fall below the ground line. When this is done, the diagonals will have a direction in the construction contrary to their true direction in space.

For example, had we so revolved the horizontal plane before finding the perspective of the point *c*, the horizontal projection of *c*, instead of being above the ground line at *c*, would have been below it at *g*, *dg* being equal to *dc*. In this case the diagonal *gf* would intersect the ground line at the point *f*, as before, and the perpendicular would meet the ground line at *d*. If then, we draw *dC'* and *fa'*, we obtain *c'*, the perspective of the point as before. But there is this difference; were we to consider the point *g* in front of the perspective plane, which we ought to do if the horizontal plane were revolved in the usual way, the diagonal *g'f* would have its vanishing point at *b'*, and the perspective of the point would be below the ground line.

We may then project points which are behind the perspective plane, on the part of the paper in front of the ground line, by observing that the true vanishing point of any diagonal is the one opposite to that which its projection indicates.

80. If through the point of sight a plane be passed parallel to any plane in space, it will contain all lines drawn through the point of sight and parallel to the latter plane. Hence, the trace of the visual plane will be the locus of the vanishing points of all lines contained in the parallel plane.

Therefore, the *vanishing line of any plane* is the trace, on the perspective plane, of the visual plane drawn parallel to it.

PROBLEM II.

Having four cubes placed in the four angles of a square, their bases in the same horizontal plane, it is required to find the perspective of the cubes and the perspective of their shadows on the plane of their bases.

81. If the perspective plane were taken through the front faces of the cubes whose horizontal projections are *cr* and *ds* (Pl. 13, Fig. *m*), AC would be its horizontal trace, and the front faces of the cubes being in the perspective plane, would be their own perspectives. The projection in this figure, although made on a small scale, shows the position which the cubes have with each other, and with the perspective plane.

Let us also suppose the point of sight to be in a plane equidistant from the inner faces of the cubes.

Draw any line, as CD, for the ground line of the plane on which the perspective is to be made.

Assume E for the centre of the picture, draw the horizontal line H′EH, and take H and H′ for the vanishing points of diagonals. Then EH, or EH′ is equal to the distance of the point of sight from the perspective plane.

Draw EE′ perpendicular to the ground line, and from E′ lay off E′*d* and E′*c*, each equal to half the distance between the inner faces of the cubes. Make *db* and *ca* each equal to the length of an edge of the given cubes, and on them construct the squares *db*′ and *ca*′; these squares are the perspectives of the faces which are in the perspective plane. Through the points *a*, *c*, *d*, *b*, *a′*, *c′*, *d* and *b′*, draw lines to E, the centre of the picture. These lines are the indefinite perspectives of the lines *a q*, *c p*, *d l*, *b h* (Fig. *m*), and of the parallel edges directly over them (72).

Through *a* and *b* draw the lines *a*H and *b*H′ to the vanishing points of diagonals; these lines are the indefinite perspectives of *a h* and *b q*, Fig. *m*.

The points *e*, *m*, *f* and *g*, in which the perspectives of the diagonals intersect the perspectives of the perpendiculars, are, respectively, the perspective of the points *e*, *m*, *f*, and *g*, Fig. *m*.

Through *e* draw *e e′* parallel to *c c′*; the point *e′*, in which it intersects *c′*E, is the perspective of the angular point of the cube which is horizontally projected at *e*, Fig. *m*. For the edge of the cube which pierces the horizontal plane at *e* is parallel to the edge which pierces it at *c*, and both are parallel to the perspective plane; hence their perspectives are parallel (73). But both these edges are limited by the edge which pierces the perspective plane at *c′*, and the indefinite perspective of this latter edge is *c′*E; therefore *e′* is the perspective of the angular point directly over the one whose perspective is *e*. Similar reasoning will apply to the other vertical edges of the cubes.

Through *e* draw *e r* parallel to *a c*; the point *r*, where it meets *a*E is the perspective of the angular point *r*, Fig. *m*. From *r* draw *r r′* parallel to *a a′*, and from the point *r′* where it intersects *a′*E draw *r′e′*.

We have then determined the square *a c c′ a′*, the perspective of the face in the perspective plane: the trapezoid *a c e r*, the perspective of the base of the cube: the trapezoid *c e e′ c′*, the perspective of a face perpendicular to the perspective plane: the square *r e e′ r′*, the perspective of the face parallel to the perspective plane; the trapezoid *c′e′r′a′*, the perspective of the upper face of the cube: and the trapezoid *arr′a′*, the perspective of a second face perpendicular to the perspective plane.

The perspective of the angular point *m* of the cube *mq*, Fig. *m*, has already been determined; and since the edge *mm*, Fig. *m*, is parallel to the perspective plane, the perspective of the point *n* must lie in *m n*, drawn through *m*, parallel to the ground line; but it is also in *a*E, the indefinite perspective of *aq*; hence it is at *n*, their point of intersection.

Through *m* and *n* draw vertical lines and produce them till they meet the lines *e′*E and *a′*E, and join the points of intersection; we have then the perspective of the face parallel to the perspective plane.

Through *n* draw *n*H to the vanishing point of diagonals; this line is the perspective of *np*, Fig. *m*, and the point *p*, where it intersects *c*E, is the perspective of the point *p*, Fig. *m*. Through *p* draw *pq* parallel to the ground line till it meets *a*E; and at *q* and *p* erect perpendiculars to the ground line, and complete the perspective of the cube as in the last case. The perspectives of the other cubes are determined in a manner entirely similar.

It remains to find the perspective of the shadows cast on the horizontal plane.

82. The shadow which a point casts upon a plane, is always found in the ray of light passing through the point, and also in the projection of this ray upon the plane on which the shadow falls. Hence, the perspective of the shadow will be found in the perspective of the ray, and in the perspective of its projection, and is consequently their point of intersection.

83. The perspective of the shadow cast by a right line

on a plane, is in the indefinite perspective of the intersection of a plane of rays passed through the line, with the plane on which the shadow falls.

84. Since the rays of light are parallel, they have a common vanishing point (71). This point is where a ray of light drawn through the point of sight pierces the perspective plane.

Let (E, E′), Fig. *n*, be the direction of a ray of light. Through the point of sight (C, C′) let a ray of light be drawn, it will pierce the perspective plane at R, which is, therefore, the vanishing point of rays.

The projections of rays on any plane are also parallel to each other, and consequently they have a common vanishing point.

Through the point of sight let a line be drawn parallel to E, the horizontal projection of a ray of light. This parallel pierces the perspective plane at F, which is, therefore, the vanishing point of the horizontal projections of rays.

Now, since the vertical plane of rays through the point of sight contains the ray of light through the point of sight, and also the line drawn parallel to the horizontal projections of rays, it follows, that its trace on the perspective plane will contain both the vanishing point of rays and the vanishing point of horizontal projections. But its trace on the perspective plane is a vertical line; hence, *the line joining the vanishing point of rays and the vanishing point of horizontal projections, is perpendicular to the ground line.*

85. We see that the vanishing point of rays and the vanishing point of horizontal projections can be found when we know the direction of the light. Reciprocally, if we know the vanishing point of rays and the vanish-

ing point of horizontal projections, we can determine the direction of the light.

For, the line joining the vanishing point of rays and the centre of the picture, will be parallel to the projections of the rays on the perspective plane. And if through the centre of the picture, the perpendicular C'D be drawn to the ground line, and produced, and DC made equal to the distance from the centre of the picture to the vanishing point of diagonals, we shall have C the projection of the point of sight on the horizontal plane. Joining this point with F', the point in which the line joining the vanishing point of rays and the vanishing point of horizontal projections intersects the ground line, and we have CF' the horizontal projection of the ray of light passing through the point of sight.

86. In finding the shadows cast by the cubes on the horizontal plane, let R be the vanishing point of rays, and P the vanishing point of horizontal projections of rays.

By considering the direction of the light, it is plain, that the edge of the cube whose perspective is *c c'* will cast a shadow on the horizontal plane. But the plane of rays through this edge is perpendicular to the horizontal plane; therefore, the shadow of the edge is parallel to the horizontal projection of the rays, and consequently P is its vanishing point. But the shadow is limited by the ray through the point whose perspective is *c'*; therefore, its perspective is limited by the perspective of this ray, which is *c'*R: consequently, *c t* is the perspective of the shadow.

The next line which casts a shadow on the horizontal plane is the edge whose perspective is *c'e'*. Since this line is perpendicular to the perspective plane, it will be parallel to the horizontal plane, and therefore its shadow, which is parallel to itself, will be perpendicular to the perspective plane; and hence, its vanishing point is at E. But t is one point of its perspective; therefore tE is its indefinite perspective. This perspective is limited by e'R drawn to the vanishing point of rays.

The shadow cast by the edge whose perspective is $r'e'$ is parallel to the line itself, and also to the perspective plane; hence its indefinite perspective is nv, drawn through n parallel to the ground line. This perspective is limited by r'R, and also by rP. The line rv is the perspective of the shadow cast by the vertical edge of the cube whose perspective is rr'. Only a small part of this shadow is seen, nearly all of it being behind the cube.

The shadows of the other cubes are found in a manner so entirely similar, as not to require particular explanations.

The faces of the cubes which are in the shade are darkened in the perspective.

PROBLEM III.

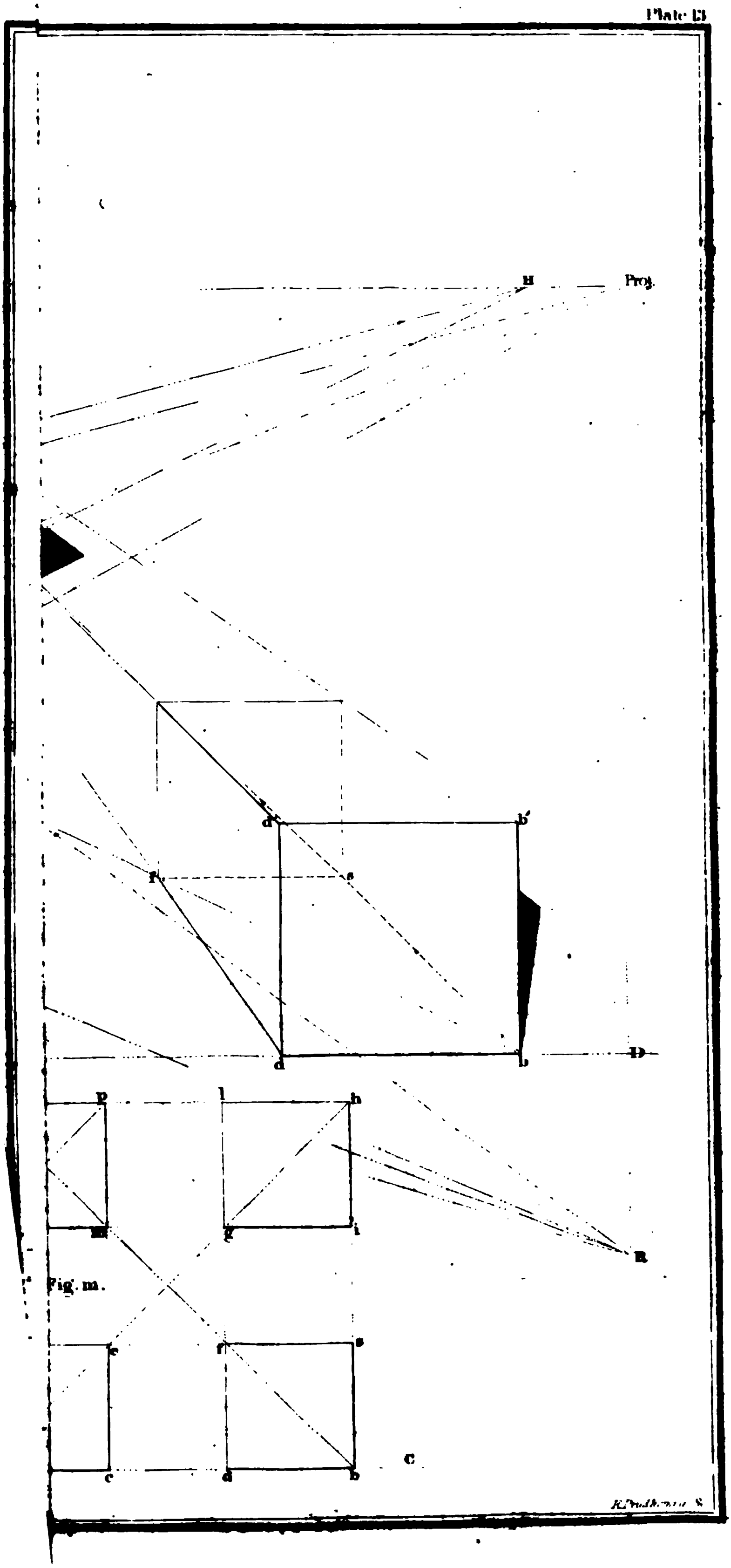
Having given four pyramids, standing on pedestals, and situated in the four angles of a square, it is required to find the perspectives of the pyramids and pedestals, and the perspectives of their shadows.

87. Fig. n (Pl. 14) represents, on a small scale, the projections of the four pyramids and pedestals, having the same relative position as those which are to be put in perspective.

Let the perspective plane be taken through the front faces of the pedestals: AB is its horizontal trace.

From any point, as a, lay off ab equal to the side of the square which forms the base of the pedestal, and on

Plate 13



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it describe a rectangle equal to the front face of the pedestal. From *b* lay off *b c* equal to the distance between the pedestals, and make *c d* equal to *a b*; and on *cd* describe a rectangle equal to the front faces of the pedestals. The perspectives of the four pedestals are then determined by constructions entirely similar to those of the last problem.

To find the perspectives of the vertices of the pyramids:

Let S be the centre of the picture, and D and D′ the vanishing points of diagonals.

If a line be drawn through the vertices of the two pyramids on the left, it will be perpendicular to the perspective plane, and will pierce it in the vertical line *h h′*, drawn through the middle point of *a b*. On this line, therefore, lay off *h h′* equal to the height of the vertices of the pyramids above the base of the pedestals, and *h′* will be the point in which the line through the vertices of the pyramids pierces the perspective plane. The line *h′*S is the perspective of this perpendicular.

The diagonal through the vertex of the front pyramid pierces the perspective plane in the horizontal line *h′a′*, and also in the vertical line *aa′*; hence *a′*D′ is the perspective of the diagonal, and v the perspective of the vertex of the front pyramid. The perspectives of the vertices of the other pyramids are easily found.

From *e* lay off *ef* equal to the distance which the pedestal projects beyond the pyramid, draw *e*D′ to the vanishing point of diagonals, and from *f* draw *f*S to the vanishing point of perpendiculars; the point *g*, in which these lines intersect, is the perspective of the point in which an edge of the pyramid pierces the upper face of the pedestal, and the horizontal line *gp* is the indefinite perspective of the line in which the front face of the pyramid intersects the upper face of the pedestal.

From *i* lay off on *i e* a distance equal to *e f*, and through the point so determined draw a line to the centre of the picture: the point *p*, in which it meets *gp*, is the perspective of the point in which a second edge of the pyramid pierces the upper face of the pedestal. The diagonals through *p* and *g* determine, by their intersections with the perpendiculars *g*S and *p*S, the points in which the two remaining edges pierce the upper face of the pedestal. Joining these points with v, the perspective of the vertex, we have the perspective of the pyramid.

Only a part of the edge of the pedestal which is perpendicular to the perspective plane at *e*, is seen. The perspective of the edge intersects the perspective of the edge of the pyramid at *l*. If through the point *l* a visual ray be drawn, it will intersect, in space, both the edge of the pyramid and the edge of the pedestal. At the point in which it intersects the edge of the pedestal, the edge of the pedestal passes behind the pyramid, and is not seen; and the same may be said of the edge of the pedestal parallel to *e i*. The perspectives of the other pedestals and pyramids are found by constructions entirely similar.

To find the perspectives of the shadows cast on the horizontal plane:

Let R be the vanishing point of rays, and P the vanishing point of horizontal projections.

The point v′, in which the diagonals *p*D and *g*D′ intersect, is the perspective of the projection of the vertex of the pyramid on the plane of the upper base of the pedestal; hence, v′P is the perspective of the projection of the ray through the vertex of the pyramid, on that plane. But vR is the perspective of the ray;

hence v'' is the perspective of the shadow cast by the vertex of the pyramid on the plane of the upper face of the pedestal.

It is plain that the edges of the pyramid which pierce the upper face of the pedestal in the points whose perspective are p and k, will cast shadows on the pedestal and on the horizontal plane. Therefore, pv'' and kv'' are the indefinite shadows on the upper face of the pedestal. It is evident, that only the parts pm and kn fall on the pedestal, and that the points m and n cast shadows on the horizontal plane.

The point r, in which the diagonals aD' and bD intersect, is the perspective of the projection of the vertex of the pyramid on the plane of the base of the pedestal. From r, draw rP to the vanishing point of projections, and the point v''', in which it intersects vR, is the perspective of the shadow cast by the vertex of the pyramid on the horizontal plane.

The line bi' is the perspective of the shadow cast on the horizontal plane by the edge bi of the pedestal, and $i'S$ is the indefinite perspective of the shadow cast by is. Therefore, drawing through m, mR to the vanishing point of rays, determines m', the shadow cast on the horizontal plane by the point m: the shadow $i'm'$ is cast by im.

The part ms, of the edge is, will not cast a shadow on the horizontal plane, being itself in the shadow of the pyramid. If, however, we draw from s a line to the vanishing point of rays, the point in which it intersects $i'S$, limits the shadow which would fall on the horizontal plane if the edge were in the light. The line drawn through s', the point so determined, and parallel to the ground line, is the indefinite perspective of the shadow cast by the edge Xs. Through n draw nR, and we determine n', the shadow cast on the horizontal plane by the point n; and $n'v'''$ is the shadow cast by the edge of the pyramid. The part ns, of the edge of the pedestal, being in the shadow of the pyramid, cannot cast a shadow on the horizontal plane; and the perspective of the shadow cast by Xn begins at n', and is parallel to the ground line.

The perspectives of the shadows of the other pyramids are found by similar constructions. The faces of the pyramids and pedestals which are in the shade and seen, are shaded in the drawing.

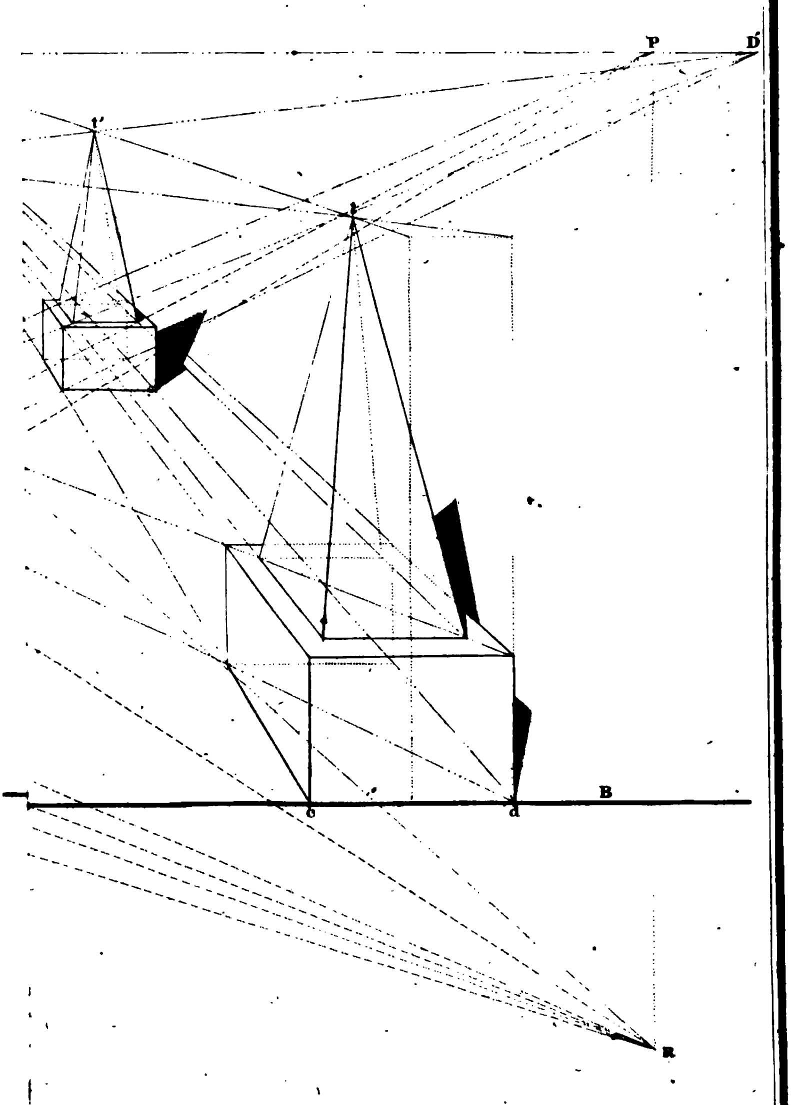
PROPOSITION IV. THEOREM.

If a right line be tangent to a curve in space, the perspective of the right line will be tangent to the perspective of the curve.

88. For let AFCG (Pl. 15, Fig. 1) be a curve, to which a right line is drawn tangent at any point as F.

Suppose a visual cone to be drawn through the curve, and a visual plane through the right line, the visual plane will be tangent to the visual cone. The perspective plane will intersect the visual plane in a right line and the visual cone in a curve, and the right line and curve will be tangent to each other (Des. Geom. 84). But the right line in which the perspective plane intersects the visual plane is the perspective of the tangent line, and the curve in which the perspective plane intersects the visual cone is the perspective of the given curve AFCG: hence, when a right line and curve are tangent in space, their perspectives are also tangent.

If two curves are tangent to each other in space, their perspectives are also tangent. For the two visual cones which determine their perspectives are tangent to each other, and therefore the curves in which they intersect the perspective plane are likewise tangent.



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PROBLEM V.

To find the perspective of a circle.

89. Let B (Pl. 15, Fig. 1), be the centre of the circle which is to be put in perspective, and RT the trace of the perspective plane; the perspective plane being perpendicular to the plane of the circle AFCG.

Although the horizontal projection of the circle is made in front of the perspective plane, all the points of it are, in fact, as far behind it as they are now projected in front of the trace RT.

Let the point of sight be taken in a plane passing through the centre of the circle, and perpendicular to the perspective and horizontal planes.

Let S be the centre of the picture, and P and P′ the vanishing points of diagonals. Through B draw the diagonal BN, and also a perpendicular to the ground line.

If the circle were in front of the perspective plane, the diagonal BN would have its vanishing point at P; but since it is behind it, the vanishing point of the diagonal is at P′ (79). Hence NP′ is the perspective of the diagonal. The perpendicular through B pierces the perspective plane at *n*, and has its vanishing point at S; therefore the point *b* where S*n* intersects the perspective of the diagonal, is the perspective of the centre B.

Through *b* draw the horizontal line *a b c*; this line is the indefinite perspective of the diameter ABC. Through A and C draw tangent lines. These tangents are perpendicular to the ground line, and their perspectives pass through S. The points *a* and *c* in which they intersect the perspective of AC, are the perspectives of the points A and C. The lines S*a* and S*c* are tangent to the ellipse, which is the perspective of the circle AFCG.

Let the perspectives of the points F and G be next found; they are *f* and *g*.

If through the points F and G tangent lines be drawn to the circle AFCG, their perspectives will be tangent to the perspective of the circle (88). But since the tangents are parallel to the ground line, their perspectives will also be parallel to the ground line; hence *gf* is perpendicular to the tangents drawn through its extremities *f* and *g*; it is, therefore, an axis of the ellipse, which is the perspective of the circle.

Bisect *gf* at *d*. Through *d* draw P*d e*, and from *e* draw the diagonal *e*D. It is plain that P*d e* is the perspective of the diagonal D*e*, and that *d* is the perspective of D.

Through D draw KDH parallel to the ground line, and find the perspectives of the points K and H, which are *k* and *h*; *k h* is the perspective of KH, and is the other axis of the ellipse. The ellipse therefore can be described.

90. When the point of sight is not in the plane passing through the centre of the given circle and perpendicular to the perspective and horizontal planes, we are unable to find the axes of the ellipse by a direct construction. We then find the perspectives of several points of the circumference of the circle, and describe the ellipse through them.

In Pl. 15, Fig. 2, the perspective of the circle is found by points. The perspectives of the tangents at the points *a*, *d*, *b*, *g*, *c* and *e* are tangent to the perspective of the circle at the points *a'*, *d'*, *b'*, *g'*, *c'* and *e'*.

91. Having given a circle in space, and the point of

sight, we may so place the perspective plane that the perspective of the circle shall be any one of the conic sections.

For, when the circle and point of sight are given, the visual cone circumscribing the circle is also given, and if the position of the perspective plane be undetermined, it may be so chosen as to intersect the cone in any one of the conic sections.

When the perspective plane is parallel to the base of the visual cone, or when it cuts the cone in a sub-contrary section, the curve of intersection is a circle.

PROBLEM VI.

To find the perspective of a cylinder, the perspective of the shadow cast by the upper circle on the interior surface, and the perspective of the shadow on the horizontal plane.

92. Let the circle described in the horizontal plane with the centre C (Pl. 15, Fig. 3), and radius CB, be the lower base of the cylinder; the centre C being at a distance behind the perspective plane equal to CC″. Let A′B′ be the projection on the perspective plane of the upper base of the cylinder, the plane of this base intersecting the perspective plane in the horizontal line A′B′.

Let the point of sight be taken in the plane through the axis of the cylinder and perpendicular to the perspective plane. Let S be the centre of the picture, and P′ and P the vanishing points of diagonals.

Find now the perspective of the lower base of the cylinder as in Prob. 5.

In finding the perspective of the upper base we have merely to regard the perpendiculars and diagonals already drawn, as the projections on the horizontal plane of corresponding perpendiculars and diagonals drawn in the upper base of the cylinder.

For example, the diagonal aBb, being considered in the upper base of the cylinder, would pierce the perspective plane at b', its perspective would be $b'a''P'$, and its intersection with $C'S$ determines a'', a point in the perspective of the upper base; the diagonal Bc determines the point c''.

If through the point of sight two tangent planes were drawn to the cylinder, they would touch it in the two elements which pierce the horizontal plane at f and g.

Having found g' and f', the perspectives of the points g and f, in the lower base of the cylinder, and g'' and f'' the perspectives of the corresponding points of the upper base, draw the lines $g'g''$ and $f'f''$; these lines are the perspectives of the elements which pierce the horizontal plane at g and f.

The part of the cylinder convex towards the point of sight, and limited by these elements, is seen; the other part is not seen. Therefore, the semi-ellipses $g'c'f'$ and $g''c''f''$, which are seen, are made full, and the semi-ellipses $g'a'f'$ and $g''a''f''$, which are not seen, are dotted.

To find the perspective of the shadow cast on the interior of the cylinder by the circumference of the upper base:

Let R be the vanishing point of rays; and H the vanishing point of horizontal projections.

If two tangent planes of rays be drawn to the cylinder in space, their horizontal traces will be tangent to the base of the cylinder; and the elements of contact will be the elements of shade. But the horizontal traces of these planes will be parallel to the horizontal projection of the rays of light; hence, their vanishing point is at H. The horizontal traces are also tangent to the base of the cylinder, therefore their perspectives will be tangent to the perspective of the base (88).

Through H draw the tangents Hk and Hh; the points of contact k and h are the perspectives of the two points in which the elements of shade pierce the horizontal plane. But since the elements of shade are vertical lines, kk' and hh', drawn perpendicular to the ground line, are their perspectives, and k' and h' are the perspectives of the points at which the shadow on the interior of the cylinder begins.

If we suppose the cylinder in space to be intersected by a plane of rays parallel to its axis, the horizontal trace of the plane will be parallel to the horizontal projection of the rays of light, and consequently, will have its vanishing point at H. Every plane so drawn will intersect the cylinder in two elements, and the one towards the source of light will cast a shadow on the other.

Through H draw any line, as Hni, to represent the perspective of the horizontal trace of a secant plane of rays. Through the points i and n, in which it intersects the perspective of the base, draw the elements ii' and nn'. Through i' draw $i'R$ to the vanishing point of rays; the point m in which it intersects nn' is the perspective of a point of shadow on the interior of the cylinder.

To find the shadow cast on any particular element, as ff'', draw from the vanishing point of horizontal projections a line, as Hf', through its foot, and through the upper extremity of the element passing through the other point in which Hf' intersects the perspective of the base, let a line be drawn to the vanishing of rays; the point p, in which it intersects the element $f'f''$, is the point of shadow required.

To find the perspective of the shadow cast on the horizontal plane:

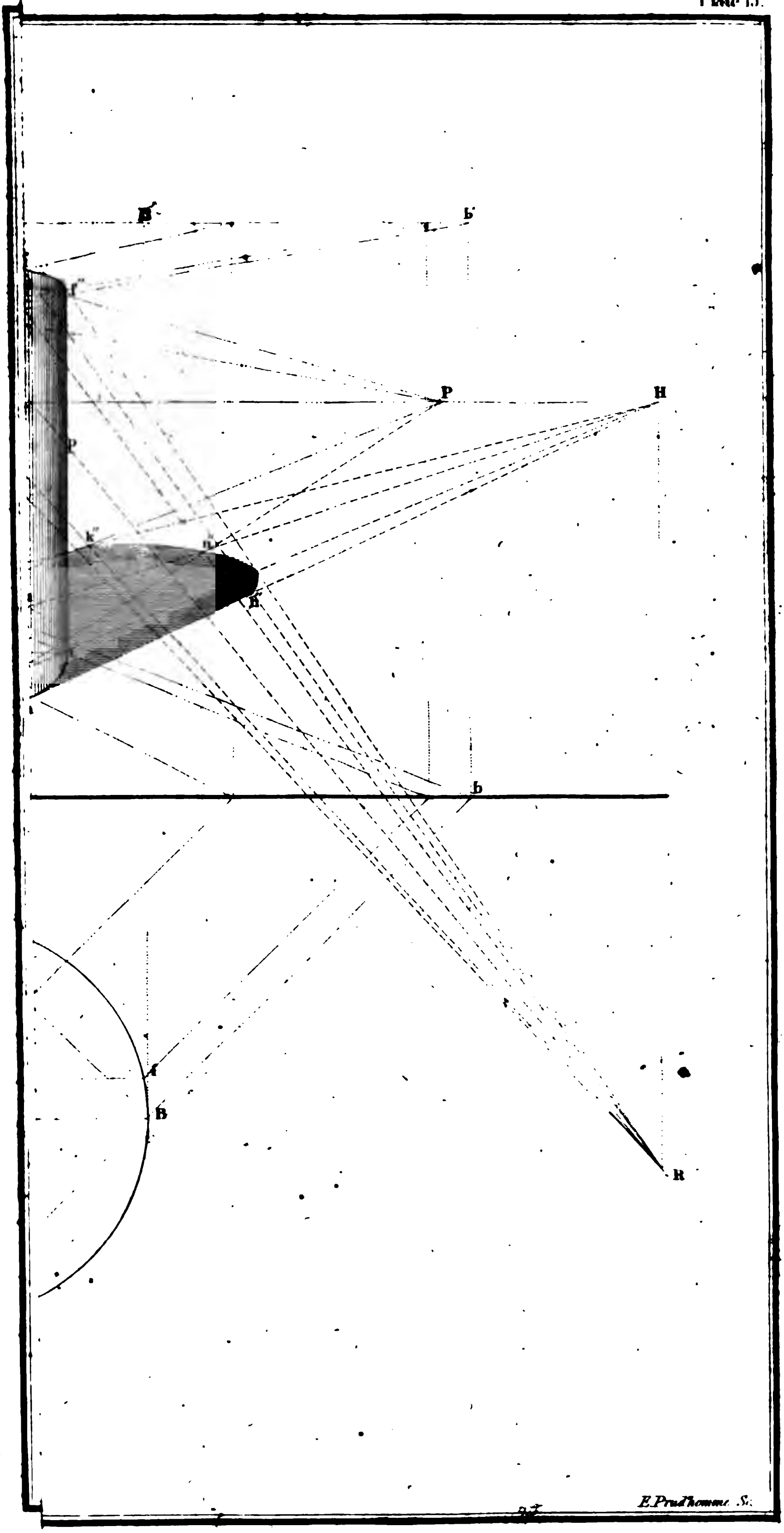
The traces of the tangent planes of rays are tangent to the shadow cast by the upper circle of the cylinder on the horizontal plane (27); therefore, their perspectives are tangent to the perspective of that shadow (88). But the rays of light passing through the upper extremities of the elements of shade, intersect the traces of the tangent planes at the points of tangency (27); therefore, k'' and h'', where the perspectives of the rays intersect the perspectives of the traces, are the perspectives of the points of tangency, and hh'', kk'' are the perspectives of the shadows cast by the elements of shade.

That part of the upper base whose perspective is $k'a''n'f''h'$, casts the curve of shadow on the horizontal plane; the remaining part casts a shadow on the interior of the cylinder.

Any line, as Hni, drawn through H, may be considered as the perspective of the horizontal trace of a plane of rays; the point n'', in which the perspective of the ray through n' intersects Hn, is the perspective of a point of shadow on the horizontal plane.

A line drawn through R, tangent to the perspective of the upper base, will also be tangent to the curve $k''n''h''$.

The part of the surface of the cylinder which is in the shade, and seen, is shaded in the drawing. The perspective of the shadow on the horizontal plane is also shaded.



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PROBLEM VII.

It is required to find the perspective of the frustum of an inverted cone; also the perspective of the shadow on the interior of the frustum, and the perspective of the shadow on the horizontal plane.

93. Let the circle described with the centre A and radius AB (Pl. 16), be the horizontal projection of the upper base of the frustum, and B′CG′ the intersection of its plane with the perspective plane. Let the circle described with the centre A and radius AH, be the lower base of the frustum, and H′I′ its vertical projection.

The horizontal projection of the vertex of the cone is at A, and its vertical projection, which is at L, is found by joining B′ and H′ and producing the line until it intersects CAE drawn perpendicular to the ground line.

Let S be the centre of the picture and D the vanishing point of diagonals.

Through (E, C), a point of the upper base, draw a diagonal and perpendicular. The diagonal pierces the perspective plane at G′, and the perpendicular pierces it at C. The diagonal has its vanishing point at D, and the perpendicular its vanishing point at S. Therefore, *e* is the perspective of the point (E,C). By similar constructions we find *e*′, the perspective of (E′,C), *c*′ the perspective of the point whose horizontal projection is *c*, and *b* the perspective of (B,B′). The perspective of any point may be found by determining the perspectives of the diagonal and perpendicular passing through it.

Before describing the ellipse *bec*′*e*′, it will be well to remember that *c*′S and B′S, being the perspectives of tangent lines to the upper base of the frustum, are tangent to the ellipse *becé* : and also, that the tangent lines in space drawn through the points (E,C) and (E′,C) are parallel to the perspective plane; hence, their perspectives are the horizontal lines drawn through *e* and *é*, tangent to the ellipse *b e c′é*.

The perspective of the lower base of the frustum is determined by constructions entirely similar.

Let the perspective of the vertex of the cone be next found.

The perpendicular through the vertex of the cone pierces the perspective plane at L, the vertical projection of the vertex; and the diagonal through the vertex pierces the perspective plane at N′; therefore the point L′, where LS intersects N′D, is the perspective of the vertex.

If through the point of sight we suppose two planes to be drawn tangent to the frustum of the cone, the traces of these planes, on the perspective plane, will pass through the perspective of the vertex, and will limit the perspective of the cone; hence, the perspectives of the upper and lower circle will be tangent to these traces. Let these tangents be then drawn through the point L′.

The point of sight being above the upper base of the frustum, the whole of that circle will be seen, and therefore its perspective is made full. A part only of the lower circle is seen, the perspective of this part is made full, and is limited by the tangent lines drawn through L′.

To find the shadow which the upper circle casts on the interior of the frustum:

Let R be the vanishing point of rays, and P the vanishing point of horizontal projections.

Since the ray of light through the vertex of the cone is a line of every plane of rays which intersects the cone in right-lined elements, the point in which this ray pierces the horizontal plane is common to the horizontal traces of all such secant planes; hence, the perspective of this point is common to the perspectives of all the traces.

But the perspective of this point is found in the perspective of the ray through the vertex of the cone, and in the perspective of the horizontal projection of this ray (82). Through R draw RL′; this line is the indefinite perspective of the ray. Through *a*′, the perspective of A, draw P*a*′; this is the indefinite perspective of the horizontal projection of the ray; the point K, in which they intersect, is the perspective of the point in which the ray through the vertex of the cone pierces the horizontal plane.

Through K, draw K*f* and K*d* tangent to the perspective of the lower base of the frustum. These tangents are the perspectives of the horizontal traces of the two planes of rays which are tangent to the frustum in space. Hence L′*fh* and L′*dg* are the perspectives of the elements of shade, and *g* and *h* the perspectives of the points at which the shadow on the interior of the frustum begins.

To find points of this shadow, draw any line through K, as K*pqs*′, which will be the perspective of the horizontal trace of a secant plane of rays. Through the points *p* and *q* draw the elements L′*pk* and L′*qs*. From *k* draw *k*R to the vanishing point of rays; the point *k*′ in which it intersects L′*s*, is the perspective of a point of shadow. The shadow on any particular element is found by drawing a line from K through its foot, and then drawing the perspective of a ray through the upper extremity of the element towards the source of light, as before.

The part of the curve whose perspective is $gbke'h$ casts a shadow on the interior of the frustum; and the part whose perspective is $hc'seg$ casts a shadow on the horizontal plane.

It is now required to find the shadow on the horizontal plane.

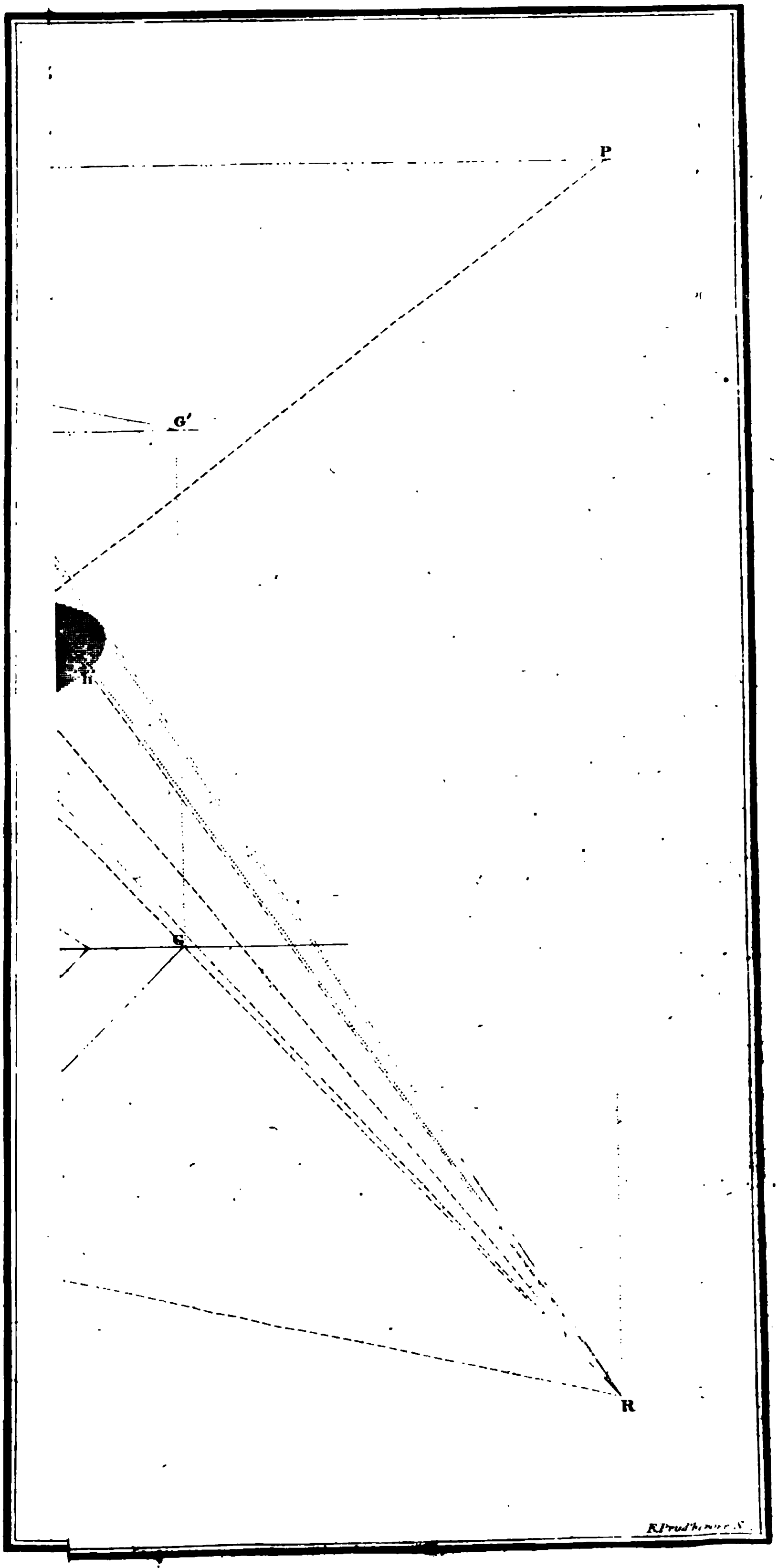
The point whose perspective is h, casts a shadow in the trace of the tangent plane of rays; but the perspective of the shadow is also in the perspective of the ray through h; hence it is at h'. For a similar reason g' is the perspective of a point of the shadow on the horizontal plane.

The line $Kpqs'$, as has already been remarked, is the perspective of the trace of a secant plane of rays. If through s, the upper extremity of the element opposite the source of light, s R be drawn to the vanishing point of rays, the point s', in which it meets $Kpqs'$, is another point of the perspective of the shadow. By similar constructions any number of points may be found.

The perspective of the shadow is tangent to the lines Kh' and Kg', at the points h' and g'. A line drawn through R, tangent to perspective of the upper base of the frustum will also be tangent to the curve $g's'k'$.

The part of the exterior surface of the frustum which is in the shade, and seen, is shaded in the drawing. We also shade that part of the interior of the frustum on which the shadow falls, and which is seen. Of the shadow which falls on the horizontal plane, all that is seen is shaded.

Plate 16.



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PROBLEM VIII.

To find the perspective of a niche, and the perspective of the shadow cast on its interior surface.

94. Let the perspective plane be taken through the front face of the niche. Let AB (Pl. 17) be the ground line, and the semicircle *b a r* the horizontal projection of the niche.

Draw *b c* and *re*, in the perspective plane, perpendicular to the ground line, and make them equal to the height of the cylindrical part of the niche; and on *ce* describe the semicircle *c l e*. These are the lines of the niche which are in the perspective plane.

The perspective of the lower base of the niche is *r a′ h b*, the segment of an ellipse: the perspective of the semicircle of contact of the cylindrical and spherical parts, is the elliptical segment *cnpte*; both of these curves are found by methods already explained. The lines *b*S and *r*S, are the perspectives of the two lines drawn tangent to the base of the niche at the points *b* and *r*; hence they are tangent to the elliptical segment *rahb*. (88.)

To find the shadow on the interior of the niche. The first line which casts a shadow on the interior of the niche, is the element *c b*. If through this element a plane of rays be passed, it will intersect the surface of the niche, in space, in a second element on which the shadow will fall. But, since the plane of rays is vertical, its horizontal trace is parallel to the horizontal projection of the rays of light; hence its vanishing point is at H, the vanishing point of horizontal projections of rays, and *b*H is its perspective.

Through d, the point in which bH intersects $rd'b$, draw dd' parallel to bc; dd' is the indefinite perspective of the element that receives the shadow. Through c, the upper extremity of the element casting the shadow, draw cR to the vanishing point of rays; the point d'', in which it intersects dd'', is the perspective of the shadow cast by the point c, and dd' is the shadow cast on the cylindrical part of the niche, by the element bc. The line bd is the perspective of the shadow which a part of the same element casts on the horizontal plane.

Through H draw any line, as Hf, near to Hb. This line may be regarded as the perspective of the horizontal trace of a vertical plane of rays; and hh' is the perspective of the element in which it intersects the cylindrical part of the niche. The plane intersects the perspective plane in the line ff'; therefore the point h', in which f'R intersects hh', is the perspective of the shadow cast by the point (f, f') on the cylindrical part of the niche.

To find the perspective of the shadow which falls on the spherical part of the niche:

If we suppose the quadrant of the sphere, which forms the spherical part of the niche, to be intersected by a plane parallel to the front face of the niche, the section will be a semicircle whose diameter will be a chord of the semicircle of contact of the cylindrical and spherical parts of the niche. The front circle of the niche will cast a shadow on this plane equal to itself (28); and the point where this shadow intersects the circle cut from the sphere is a point of the required shadow in space. If then, we find the perspectives of these two circles, the point in which they intersect will be the perspective of a point of the curve of shadow.

Since both the circles are parallel to the perspective plane, their perspectives will be circles (91). Draw any line, as np, to represent the perspective of the diameter of the semicircle cut from the spherical part of the niche by the parallel plane, and on it describe the semicircle nkp. The perspective of the shadow cast on the plane of this circle by the centre g, is found in gR, and also in the perspective of the projection of the ray through g on that plane. But, since the plane is parallel to the perspective plane, the projection of the ray upon it is parallel to SR. But g' is the perspective of one point of its projection; hence $g'g''$, drawn parallel to SR, is the perspective of the projection of the ray on the parallel plane, and g'' is the perspective of the shadow cast by the centre g. The shadow cast by cg is parallel to itself and to the perspective plane; hence $g''q$, drawn parallel to cg, is the perspective of the indefinite shadow cast by cg on the parallel plane. But this shadow is limited by the ray cR; hence $g''q$ is the perspective of the radius of the circle of shadow. With g'' as a centre, and the radius $g''q$, describe the arc qk; the point k, in which it intersects nkp, is the perspective of a point of the curve of shadow.

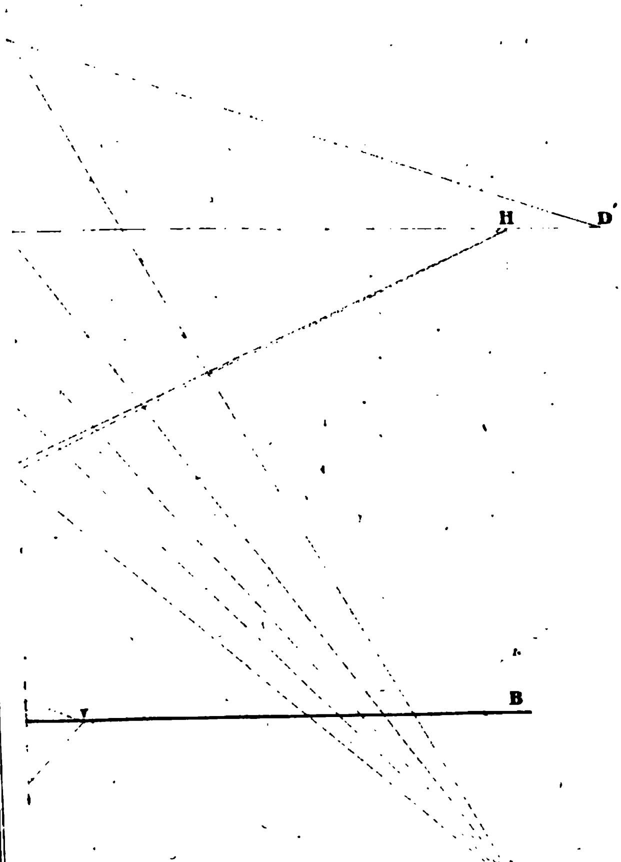
If through the centre g, gi be drawn perpendicular to SR, it will be a line of the perspective plane, and of the plane of the circle of shadow; therefore the point i, in which it meets the circumference cle, is a point of the perspective of the curve of shadow.

We can easily find the perspective of the point at which the shadow passes from the spherical to the cylindrical part of the niche. For, if through the point in space of which k is the perspective, a line be supposed drawn parallel to gi, it will be contained in the plane of shadow, and will pierce the upper base of the cylinder in the trace of the plane of shadow. But ks, drawn parallel to gi, is the perspective of this parallel, and s is the perspective of the point in which it pierces the upper base of the cylinder. Hence, gst is the perspective of the trace of the plane of shadow, and t the perspective of the point at which the shadow passes from the spherical to the cylindrical surface.

There are other constructions for finding the shadow on the spherical part of the niche.

Revolve the plane of rays passing through the point of sight and perpendicular to the perspective plane, about SR, until it coincides with the perspective plane. The point of sight falls in SD$''$, drawn perpendicular to SR, and at a distance from S equal to SD (76). The ray of light through the point of sight takes the position RD$''$.

Intersect the spherical part of the niche by a plane of rays perpendicular to the vertical plane. Let $lm'z$, drawn perpendicular to gi, or parallel to SR, be the trace of such a plane. Revolve this plane until it coincides with the perspective plane. The semicircle cut out of the sphere, when revolved, is the semicircle lmg, and the ray through l takes the position lm, parallel to RD$''$. Let the counter revolution be now made, and draw m'S, which is the perspective of mm', and lR, which is the perspective of the ray; the point m'', in which they intersect, is the perspective of a point of the curve of shadow. Other points may be found by similar constructions.



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PROBLEM IX.

To find the perspective of a sphere, the perspective of its shade, and the perspective of its shadow on a horizontal plane.

95. If we suppose the sphere to be circumscribed by a visual cone, the curve in which the perspective plane intersects the cone will be the perspective of the sphere. The point of sight and the place of the sphere being given, the perspective plane may be taken in such a position as to intersect the visual cone in any of the conic sections.

Although it is easy to understand why the perspective of a sphere may be an ellipse, a parabola, or an hyperbola; yet neither of these curves seems to be a proper representation of a body perfectly round. It is for this reason that the perspective of a sphere is generally drawn on a plane perpendicular to the line joining the point of sight and its centre; when the perspective plane has this position, it intersects the visual cone in a circle.

Let AB (Pl. 18), be the ground line, S the centre of the picture, D the vanishing point of diagonals, R the vanishing point of rays, and H the vanishing point of horizontal projections of rays.

Suppose the centre of the sphere to be at a distance behind the perspective plane, equal to its radius.

Let S be the projection of its centre, and the circle described with the radius SC the projection of the sphere on the perspective plane.

If through the point of sight we suppose a plane to be passed perpendicular to the horizontal and perspective planes, LSa' will be its trace on the perspective plane. This plane will intersect the sphere in a great circle, and the visual cone in two elements which will be tangent to it. Let this plane be revolved about its trace LSa', until it coincides with the perspective plane. The point of sight will fall at D (76), and the centre of the circle cut out of the sphere at C. With C as a centre, and radius CS, describe the semicircle, as in the figure; and then draw from D the tangent line D$a'a$, which will be, in its revolved position, an element of the visual cone. This element pierces the perspective plane at a'; hence, the circle described with the radius Sa', is the perspective of the sphere.

It is now required to find the perspective of the curve of shade.

It will first be necessary to find the vanishing line of the plane of the circle of shade (80).

Since the plane of shade is perpendicular to the direction of the light (34), the required line is the trace of a plane passing through the point of sight and perpendicular to the rays.

Through the point of sight let a plane of rays be passed perpendicular to the perspective plane; RSE is its trace. Let this plane be revolved about RE, to coincide with the perspective plane. The point of sight falls at F, in a perpendicular to RS, and at a distance from S equal to SD.

Since the ray through the point of sight pierces the perspective plane at R, it will, after the revolution, take the position RF. If then FE be drawn perpendicular to RF, it will be a line of the plane through the point of sight, and perpendicular to the direction of the ray, and E will be a point of its trace. But RE is the projection, on the perspective plane, of the ray passing through the point of sight (85): and since the plane through the point of sight is perpendicular to this ray, the trace will be perpendicular to its projection. Therefore, GEI, drawn perpendicular to RE, is the vanishing line of the plane of shade.

If we suppose two planes of rays to be drawn tangent to the visual cone which determines the perspective of the sphere, they will also be tangent to the sphere in space, and the points of contact will be points of the curve of shade. The perspective of these points of contact will be found in the perspective of the sphere, and in the traces of the tangent planes. Since the tangent planes are planes of rays, they will contain the ray through the point of sight; hence their traces will pass through the point R. But these traces must also be tangent to the perspective of the sphere; hence R*b* and R*c*, drawn tangent to the perspective of the sphere, are the traces of the tangent planes of rays, and *b* and *c* are two points of the perspective of the circle of shade. And since S is the perspective of the centre of the sphere, the lines *b*S*d* and *c*S*e* are the indefinite perspectives of two diameters of the circle of shade.

Let us suppose, for a moment, the circle of shade to be represented by the circle *dcbq*, Fig. *n*, and let *bd* and *c e* be the diameters already referred to. The traces of the tangent planes of rays are the tangents *nb* and *nc*; and since the tangent planes of rays are perpendicular to the plane of shade, they will intersect in a ray of light perpendicular to the plane of the circle *pbqc* at *n*. If through this ray and the centre *a*, a plane be passed, its trace *nq* will bisect the angle *cab* and be parallel to the lines *b e* and *c d*, joining the corresponding extremities of the diameters *bd* and *ce*. Hence, the plane of rays whose trace is RE, intersects the plane of the circle of shade in a line making equal angles with the diameters whose perspectives are *bd* and *ce*. But this line is parallel to the chords joining the corresponding extremities of these diameters, and also to the line FE, in which the plane of rays RE intersects the parallel plane through the point of sight. Therefore, E is the common vanishing point of the trace on the plane of shade, and of the parallel chords joining the corresponding extremities of the diameters. Draw *b*E and *c*E. The points *e* and *d* in which they intersect *c*S*e* and *b*S*d*, are the perspectives of the extremities of these diameters, and consequently, of two more points of the curve of shade,

To find other points of the curve of shade:

Let the plane passing through the point of sight and parallel to the plane of shade be revolved about its trace GI, to coincide with the perspective plane. The point of sight falls at F′, a distance from E equal to EF, Through this point let any two lines, as F′G and F′I, be drawn at right angles to each other, and note the points I and G in which they meet the trace GI. Let us now consider the plane to be revolved back to its position in space.

If through the two extremities of any diameter of the circle of shade, two lines be drawn parallel to F′G and F′I, they will be contained in the plane of shade, and their point of intersection on the surface of the sphere will be a point of the circle of shade. But these two lines will have G and I for their vanishing points (71). If therefore, through the points *d* and *b*, we draw the lines *d*G and *b*I, they will be the perspectives of two lines drawn through the extremities of a diameter at right angles to each other, and the point *g*, in which they intersect, is the perspective of a point of the circle of shade. The lines through the points *c* and *e* determine the point *f*.

If through *f*, we draw *f*S*h*, it will be the indefinite perspective of a diameter of the circle of shade. Through *e* draw *e*I. This line is the perspective of the chord parallel to the chord whose perspective is *fe*; therefore, *h* is the perspective of the other extremity of the diameter, and consequently, the perspective of a point of the circle of shade.

Having found a sufficient number of points of the curve of shade, let it be described. Only the part *cfgb*, which is in front of the circle of contact of the visual cone and sphere, is seen.

It is now required to find the perspective of the shadow cast on the horizontal plane.

It will first be necessary to find the perspective of the horizontal trace of the plane of shade.

Since this trace is a horizontal line, its vanishing point is in the line TSH (74), and since it is a line of the plane of shade, its vanishing point is in the line GI; hence it is at T.

It is necessary in the next place to find the perspective of the horizontal projection of the centre of the sphere. To do this, lay off from L to P the radius SC of the sphere; P is the point at which the diagonal through the horizontal projection of the centre of the sphere pierces the perspective plane, and the perpendicular through the same point pierces it at L; hence *i* is the perspective of the horizontal projection of the centre of the sphere.

We will here premise, in order to illustrate what follows, that when a line intersects the perpendicular from the point of sight to the perspective plane, its perspective and its projection on the perspective plane are the same line: for, the visual plane which determines its perspective is then perpendicular to the perspective plane.

A second point in the perspective of the horizontal trace of the plane of shade is found, by finding the perspective of the point in which any diameter of the circle of shade pierces the horizontal plane. To simplify the construction, we will take that diameter which is parallel to the perspective plane.

Since the diameter is perpendicular to the ray through the centre of the sphere, and since the perspectives of the two lines are the same as their projections, it follows, that their perspectives will be at right angles to each other (Des. Geom. 51). But SR is the perspective of the ray; hence, SF drawn perpendicular to SR, is the indefinite perspective of the diameter. The projection of this diameter on the horizontal plane, passes through the horizontal projection of the centre of the sphere, and is parallel to the ground line; hence, its perspective passes through *i* and is parallel to AB (72). Therefore, *k* is the perspective of the point in which the diameter pierces the horizontal plane, and consequently, a point in the perspective of the horizontal trace of the plane of shade; and T*k* is the perspective of that trace.

The line *t*H is the perspective of the horizontal projection of the ray through the centre of the sphere, and SR is the perspective of the ray; hence *p*, their point of intersection, is the perspective of the shadow cast on the horizontal plane by the centre of the sphere.

The shadow cast on the horizontal plane by any diameter of the circle of shade, will pass through the point in which the diameter pierces the horizontal plane, and also through the point of which *p* is the perspective. Therefore, produce the diameter *bd* till it meets T*q*, the perspective of the horizontal trace of the plane of shade, and from *r* draw *rd'pb'*; this line is the perspective of the indefinite shadow cast by the diameter on the horizontal plane. Through the extremities b and d of the diameter, draw lines to R; the points b' and d' are points of the perspective of the shadow. The diameter ec being produced to q, gives the points c' and e'. The perspective of the shadow will be tangent to the lines Rb and Rc at the points b' and c'.

96. We can find, by a direct construction, the axes of the ellipse, which is the perspective of the circle of shade.

That the figure may not become too complicated, we will make the construction in Fig. *m*, in which the projection and perspective of the sphere are both represented, and in which R is the vanishing point of rays.

If through the axis of a scalene cone, having a circular base, a plane be passed perpendicular to the base, it will divide the cone into two symmetrical parts. If then, a plane be passed perpendicular to the plane through the axis, it will intersect the cone in a curve whose axis is the intersection of the two planes. The second axis of the curve is the line drawn through the middle point of the first, and perpendicular to it.

The visual cone, which is formed by drawing visual rays to all the points of the circle of shade, is a scalene cone with a circular base. The plane of rays, whose trace is SR, passes through the axis, is perpendicular to the plane of shade, and also to the perspective plane. Hence, the line RS, in which it intersects the perspective plane, contains an axis of the ellipse in which the perspective plane intersects the visual cone, or an axis of the ellipse which is the perspective of the circle of shade. The plane, whose trace is RS, also intersects the plane of the circle of shade, in a diameter perpendicular to the direction of the light. Let this plane be revolved about RS to coincide with the perspective plane. The point of sight falls at F; RF is the revolved position of the ray; the centre of the sphere falls at s, and psq, drawn perpendicular to RF, is the diameter of the circle of shade. Through p and q draw Fp and Fq; $p'q'$ is evidently the perspective of pq, and is an axis of the ellipse. Through a, the middle point of $p'q$, draw Faa'. The chord of the circle of shade, whose perspective is the other axis of the ellipse, passes through the point a', in its true position in space, and is perpendicular to $pa'q$. But since the circle of shade is a great circle of the sphere, the length of the chord is equal to $ca'b$. Let $ca'b$ be revolved about the middle point a', till it has the position $c'a'b'$, parallel to fR. Through F draw Fc' and Fb'; the points f and d, in which they intersect fR, limit the other axis of the ellipse. But this axis must pass through a, and be perpendicular to $p'q'$. Therefore, drawing the perpendicular $f'd'$, and making ad' and af' equal to af, or ad, we have the second axis of the curve.

PROBLEM X.

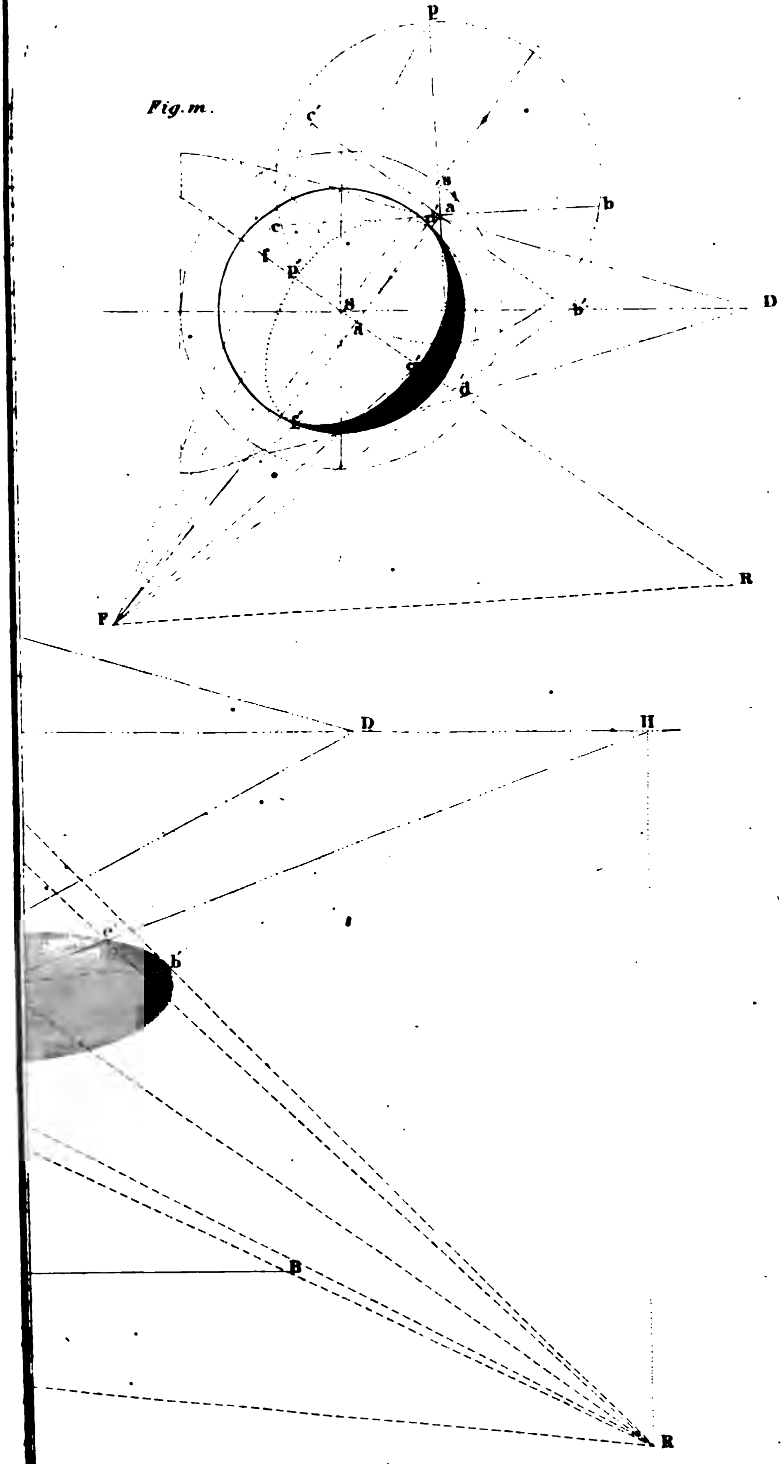
To find the perspective of the groined arch and the perspective of its shadows.

97. The arch is supported by four pillars, capped by cornices, and standing on pedestals; the pedestals are placed in the four angles of a square. The arch itself is formed by portions of two equal cylinders.

Fig. n (Pl. 19) represents the two projections of the pedestals, the pillars, the cornices, and the arch. On the line (ab, $a'b'$), joining the points in which the inner front edges of the pillars, produced, pierce the upper plane of the cornices, let the semicircle $a'sb'$ be described, its plane

Plate 18

Fig. m.



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coinciding with that of the front faces of the pillars; and on *de* let there be described an equal and parallel semicircle.

If now a right line be moved from the position (*ad*, *a*′), parallel to itself, and touching the two semicircles, it will generate a cylinder whose axis is (*cm*, *c*′), and whose elements are perpendicular to the front face of the arch. On the two lines *gf* and *rv*, joining the points in which the edges of the pillars pierce the upper plane of the cornices, let two equal and parallel semicircles be described, and let a right line be moved along them, parallel to itself, generating a semi-cylinder whose axis *zx* is parallel to the front face of the arch, or perpendicular to its side faces.

The two cylinders which have been generated are equal; their axes are at right angles to each other, and the surfaces of the cylinders intersect in two equal semi-ellipses, called the groins of the arch.

These groins spring from the upper plane of the cornices, the one from *n* to *q*, the other from *l* to *t*. They intersect each other in a point of which *o* is the horizontal projection, and which is at a distance above the plane of the cornices equal to the radius *c*′*s*.

In the construction of the arch, only a part of each cylinder is used. The parts *alonb* and *dqote*, are formed by the cylinder whose axis is *cm*; and the parts *rloqv* and *gnotf*, by the cylinder whose axis is *zx*; the other parts of both cylinders are supposed to be removed.

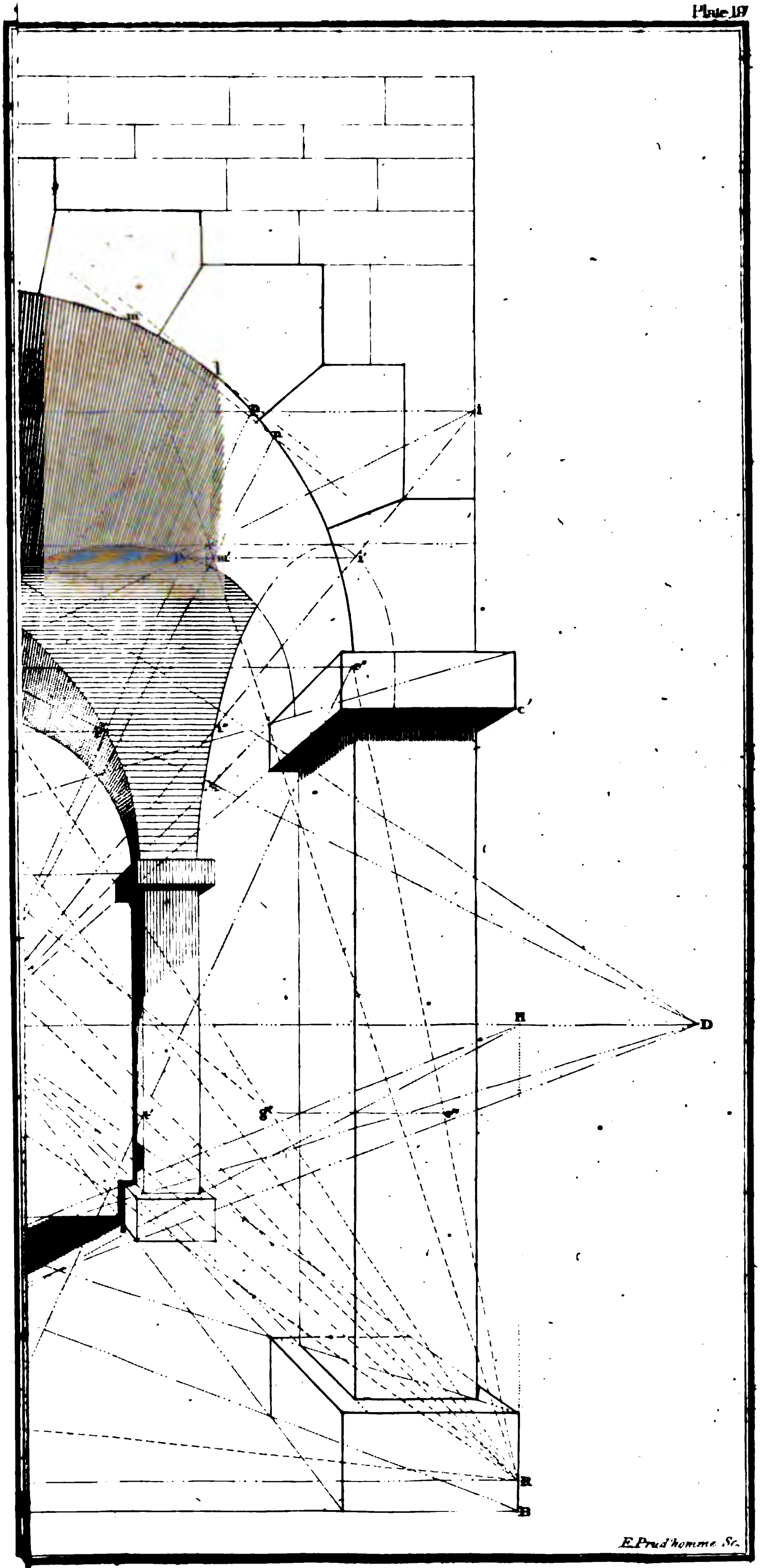
If we suppose the outer planes of the pillars to be produced above the upper plane of the cornices, they will form vertical faces of a prism, whose horizontal sections are squares. We shall call this prism the solid part of the arch.

Let the perspective plane be taken to coincide with the front faces of the pedestals and cornices, and let AB be the ground line. Let S be the centre of the picture, D′ and D the vanishing points of diagonals, R the vanishing point of rays, and H the vanishing point of horizontal projections. From A and B lay off distances equal to a side of the squares which form the bases of the pedestals, and on them describe rectangles equal to the front faces of the pedestals. Then, let the perspectives of the pedestals be found as in Prob. 2.

From *a* lay off *ab* equal to the distance which the pedestal projects beyond the pillar. From *a* draw *a*D to the vanishing point of diagonals, and from *b* draw *b*S to the vanishing point of perpendiculars; *d* is the perspective of the point in which an edge of the pillar pierces the pedestal: and the horizontal line through *d* is the indefinite perspective of the line in which the front face of the pillar intersects the pedestal.

From the other extremity of the line *ab* lay off a distance equal to *ab*, and draw a line to S; the point *e*, in which it intersects *de*, is the perspective of the point in which a second edge of the pillar pierces the pedestal, and the point in which it intersects the diagonal *a*D, is the perspective of the point in which a third edge pierces the pedestal. Through this latter point draw a parallel to *de*; the point in which it intersects *b*S is the perspective of the point in which the fourth edge of the pillar pierces the pedestal. The four vertical lines drawn through these points are the indefinite perspectives of the vertical edges of the pillar. The front and inner side faces only are seen.

From *a* draw a vertical line in the perspective plane, and on it lay off *ac* equal to the distance between the upper plane of the pedestals and the lower plane of the cornices, and through *c* draw an indefinite horizontal



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line cc'. Since the front faces of the cornices are in the perspective plane, we can lay off the thickness of the cornices cc'', and their width cf, equal to the width of the pedestals. Project the point b into the lower plane of the cornices at b', and through b' draw b'S. The point in which this line intersects the edge of the column through d, is the perspective of the point in which that edge pierces the lower plane of the cornice; and the horizontal line through this point is the perspective of the line in which the front face of the column intersects the lower plane of the cornice. From the point in which this horizontal line intersects the edge of the column through e, draw a line to S; this line is the perspective of the intersection of the inner side face of the column with the cornice. The line drawn from c to S is the indefinite perspective of the edge of the cornice which is perpendicular to the perspective plane at c; the part which is seen is limited by the edge of the pillar.

From f, draw fS, and from c, a diagonal cD; their point of intersection f', is the perspective of the angle of the cornice, diagonally opposite to c. Through f' draw a vertical line, and from the upper extremity of the perpendicular through f, draw a line to S; their point of intersection is the perspective of an angular point of the cornice. The horizontal line through f' is the perspective of the edge parallel to cf. We have now found the perspective of one pillar and one cornice. The perspectives of the others are found by similar constructions.

We will next find the perspectives of the front and back circles of the arch.

Find e' and e'', the perspectives of the points in which the front and inner edges of the pillars pierce the upper planes of the cornices; draw $e'e''$, and on it describe a semicircle, making that part full which is above the upper lines of the cornices; this circle is the perspective of the front circle of the arch.

Find the perspectives of the points in which the edges of the pillars that are projected at *d* and *e* (Fig. *n*), pierce the upper plane of the cornices; the semicircle described on the line joining these points, is the perspective of the back circle.

We next find the perspectives of the groins, and the perspectives of the side circles.

First, find the perspectives of the points *n*, *q*, *l* and *t* (Fig. *n*), in the upper plane of the cornices, from which the groins spring; and also the perspectives of the points *g*, *f*, *r* and *v*, in the same plane, from which the side circles spring.

Let us now suppose the solid part of the arch to be intersected by a horizontal plane, and let *hi* (Fig. *n*), be its trace on the front face of the arch. This plane will intersect the solid part of the arch in a square. It will intersect the cylinder whose axis is *cm*, in two elements perpendicular to the front face of the arch at the points *k* and *p*. If through the points in which the horizontal projection of either element intersects the lines *nq* and *lt*, lines be drawn parallel to the ground line, they will be the horizontal projections of the elements in which the secant plane intersects the other cylinder. The points p', p'', k' and k'', are in the groins, and the square $k'p'p''k''$ has the same diagonals with the square in which the secant plane intersects the solid part of the arch.

Draw any horizontal line, as *hi*, for the perspective of the trace of such a secant plane. From *h* draw *h*S and *h*D; from *i* draw *i*S and *i*D′; and from the points *k* and *p*, draw *k*S and *p*S. The points k' and p'' are the perspectives of the points k' and p'' (Fig. *n*), of the groin

lt; and the points p' and k'', the perspectives of the points p' and k'' of the groin nq.

The line $k'p'$ is the perspective of the element $k'p'$ (Fig. n), and therefore the points h' and i', in which it intersects hS and iS, are the perspectives of points h' and i' (Fig. n), of the side circles. If we draw a line through the points k'' and p'', the points h'' and i'', in which it intersects hS and iS, are the perspectives of the two points h'' and i'' (Fig. n), of the side circles.

If we use a second secant plane, we determine two other points in the perspective of each groin, and two points in the perspective of each of the side circles.

If through the point of sight a plane be passed tan gent to the cylinder whose elements are parallel to the perspective plane, it is evident that the perspective of the element of contact will pass through the highest point of each of the curves. But the perspective of this element is a horizontal line (72); hence it will be tangent to each of the curves.

To find this element, and its perspective, suppose a vertical plane to be passed through the point of sight, and perpendicular to the perspective plane. This plane, whose trace is R$'g$, will intersect the cylinder in a circle whose centre is in the upper plane of the cornices, and equally distant from the front and back faces of the arch. Let this plane be revolved about R$'g$, to coincide with the perspective plane. The point of sight falls at D, and the centre of the circle at c''. With c'' as a centre, and $c''a''$, equal to a line corresponding to oa (Fig. n), describe the arc of a circle, and from D draw a tangent to this arc. This tangent is a line of the tangent plane, and the point in which it meets Sg, produced, is the point at which it pierces the perspective plane. The horizontal line drawn through this point is the perspective of the element to which the curves are tangent. Let the curves be now described.

The points at which the perspectives of the groins intersect the perspectives of the side circles, will limit the parts of the side circles which are seen.

It is now required to find the perspective of the shadows cast by the different parts of the arch.

The front circle of the arch casts a shadow on that part of the arch formed by the cylinder whose elements are perpendicular to the perspective plane. In finding the perspective of this shadow, we have merely to find the perspective of the shadow cast by the base of a cylinder on the interior surface.

Draw through g, gl perpendicular to SR; l is the perspective of the point at which the shadow on the interior of the cylinder begins; for the tangent line at l is parallel to SR, and is the perspective of the trace of a plane of rays tangent to the cylinder. To find other points of the shadow, draw any line, as mn, parallel to the tangent at l. This line is the perspective of the trace of a plane of rays on the front face of the arch. This plane intersects the cylinder in two elements; one casts, and the other receives the shadow. The line nS is the perspective of the element which receives the shadow, and m', where it is intersected by mR, is the point of shadow. Let the curve of shadow from l be then described.

The rays of light may have such a direction, that the shadow of the front circle will intersect the groin $k'p''$ When this occurs, one curve of shadow will pass from the point of intersection along the cylinder whose elements are parallel to the perspective plane, and will pass off on the side circle $i''i''$. Another curve will pass from the point of intersection, along the cylinder whose elements are perpendicular to the perspective plane, and will pass off on the back circle of the arch.

The side circle of the arch, on the left, casts a shadow on the interior of the arch which falls on the cylinder whose elements are parallel to the perspective plane. If we suppose a tangent plane of rays to be drawn to this cylinder, the point in which the element of contact meets the end circle towards the source of light, will be the point at which the shadow on the interior of the cylinder begins. But this tangent plane will be perpendicular to the side face of the arch; hence, its trace will be parallel to the projections of rays on the side planes. If we suppose a plane to be passed through R parallel to the side planes of the arch, and the ray through the point of sight to be projected on this plane, the projection will be parallel to the projection of rays on side planes, and consequently, to the trace of the tangent plane. If a line be drawn through the point of sight parallel to the projection of the ray, it will pierce the perspective plane at R′, since RR′ is the trace, on the perspective plane, of the plane which projects the ray on the plane through R; hence R′ is the vanishing point of the projections of rays on the side planes.

If, therefore, through R′ we draw Rq, tangent to the curve $h'qh''$, it will be the perspective of the trace of the tangent plane of rays, and q is the perspective of the point at which the shadow on the interior of the arch begins.

If through R′ we draw a secant line, we may regard it as the perspective of the trace of a plane of rays, parallel to the tangent plane, and intersecting the cylinder in two elements.

If through the lower point, in which this line cuts the curve $h'qh''$, we draw a horizontal line, it will be the perspective of the element which receives the shadow; and drawing through the upper point a line to the vanishing point of rays, determines *o*, the perspective of a point of the curve of shadow. Let the curve of shadow be then drawn.

We are next to find the shadows of the cornices on the front faces of the pillars.

Through *c* draw a line to S, and another to R. The point where the line to S meets the line in which the front face of the pillar intersects the lower plane of the cornice, is the perspective of one point in the projection of the ray through *c*; on the front face of the pillar, and the line drawn through this point, parallel to SR, is the perspective of the projection: the point in which this perspective intersects *c*R is the perspective of one point of the line of shadow: but the line of shadow is horizontal both in space and in perspective.

To find the shadow cast by the column. The edge *e r* casts a line of shadow on the pedestal and on the horizontal plane, both of which are parallel to the horizontal projection of rays. Hence, their perspectives are easily found. At *s* the shadow falls on the inner side face of the back pedestal. It passes up the face in a vertical line; reaching the upper face it becomes parallel to the horizontal projections of rays, and when it reaches the face of the pillar, it ascends in a vertical line along the face. If through *r*, the highest point of the edge of the pillar which is in the light, a line be drawn to R, it will limit the shadow cast by the edge of the pillar. From the point in which this line intersects the vertical line before drawn, on the face of the pillar, the shadow will be cast by the lower line of the cornice. This shadow will be parallel to the projections of rays on the side planes, will therefore have its vanishing point at R′, and will be limited by the ray through *f*. The vertical line through f will then cast its shadow, which will be a vertical line, and will be limited by the ray drawn through the upper extremity of the vertical edge of the cornice through f. There is a small part of the edge of the cornice, parallel to ff', which is in the light, and which will cast a shadow on the face of the pillar. Its shadow will be perpendicular to the perspective plane, and consequently have its vanishing point at S.

The front circle of the arch will next cast a shadow on the side face.

To find this shadow we will observe, that the shadow of the diameter $e'e''$, on the side face, is parallel to the projections of rays on the side face, and since e'' is the perspective of the point in which the diameter pierces the plane of the side face, $e''R'$ will be the indefinite perspective of its shadow. Draw any ordinate of the circle, as tu, which it is supposed will cast a shadow. Through t draw tR; the point t' is the shadow cast on the side plane by the foot of the ordinate. But the ordinate being parallel to the side plane, its shadow will be the vertical line through t'. The point in which this vertical line intersects the ray through u is a point of shadow on the side plane. When this shadow reaches the front face of the pillar, it then falls on that face, and is a circle both in space and in perspective.

Through g draw gR; and through g', the perspective of the point at which the centre of the front circle is projected on the plane of the front faces of the back pillars, draw $g'g''$ parallel to SR; g'' is the perspective of the shadow cast by the centre of the front circle on the plane of the front faces of the back pillars. But the shadow cast by the radius ge'' is parallel to itself; therefore, draw $g''e'''$ parallel to the ground line, and $e''R$ to the vanishing point of rays; $g''e'''$ is the perspective of the radius of the circle of shadow. With g'' as a centre, and $g''e''$ as a radius, describe the arc of a circle, and we have the shadow cast on the pillar. The parts of the arch which are in the shade, or on which shadows fall, are shaded.

PROBLEM XI.

It is required to find the perspective of a house and the perspective of its shadows.

98. Having measured all the lines of the house which are to be put in perspective, make its horizontal projection to any convenient scale, as in Pl. 20.

The outer lines are the horizontal projections of the outer lines of the eave-trough, and the adjacent inner ones are the lines in which the outer faces of the walls intersect the horizontal plane. The lines in which the roofs intersect are also made, as well as the projections of the chimneys and steps.

In selecting the position of the perspective plane, reference should be had to the part of the building which is to appear most prominent in the picture. Were it only required to represent the front of the building, we should take the perspective plane parallel to it; but when the front and an end are to be represented, the perspective plane must be taken oblique to them both. It simplifies the construction, without affecting the generality of the method, to assume the perspective plane through one or two of the prominent vertical lines. Such lines being in the perspective plane will be their own perspectives, and all the vertical distances may be laid off on them.

In the construction here given, the perspective plane is passed through the vertical corners of the house which pierce the horizontal plane at B and A, and DE is the ground line. Having chosen the position of the eye, lay off the distance FS equal to its height above the ground, and on the line FG, perpendicular to DE, lay off the distance of the point of sight from the perspective plane. The horizontal projection of the point of sight does not fall on the paper, the eye being at about one and a half times the length of the building from the perspective plane. In making the drawing, a paper must be taken of such dimensions as will enable us to represent upon it the horizontal projection of the point of sight, and the vanishing points of the principal lines.

Although the horizontal projection of the house is made in front of the ground line, yet the building is in fact behind the perspective plane, with its front towards the eye. To represent to the mind the horizontal projection of the house in its true position, we must conceive the horizontal plane to be revolved 180° around the ground line DE, which will bring every point now in front of the ground line at an equal distance behind it.

At the points a and d draw aba' and dcd' perpendicular to the ground line DE, and make $b\,a'$ equal to $a\,b$, and $c\,d'$ equal to $c\,d$. The lines Aa' and Ad' are the lines Aa and Ad in the positions from which their perspectives are taken.

Through the point of sight let a line be drawn parallel to Ad': its horizontal projection will pass through the horizontal projection of the point of sight, and the line will pierce the perspective plane in a point of the horizontal line KSH. This point is the vanishing point of all lines parallel to Ad. The point is not on the paper, but as we have frequent occasion to refer to it, we will designate it by the letter K. Through the point of sight let a line be drawn parallel to A*d*: this line pierces the perspective plane at H, which is, therefore, the vanishing point of all lines parallel to A*a*.

From A draw a line to the vanishing point K; it will be the indefinite perspective of the line A*d*. From *e* draw a perpendicular to the ground line, and from the foot of the perpendicular draw a line to S; the point *e'*, in which it intersects the line AK, is one point in the perspective of the corner of the house, which pierces the horizontal plane at *e*; the vertical line drawn through the point *e'* is the indefinite perspective of the corner, and the line drawn to B is the perspective of the line B*e*.

The line drawn from B to the vanishing point K, is the indefinite perspective of the line B*f*. From *f*, draw a perpendicular to the ground line, and from the foot of the perpendicular draw a line to S; the point *f'* in which it meets the line BK, before drawn, is the perspective of the point *f*. Through the point *f'*, draw a vertical line, and it will be the indefinite perspective of the corner of the house which pierces the horizontal plane at *f*. The perspective of the point *d* is in the line AK, and also in the line joining *c* and S; hence it is at *d'* their point of intersection. The vertical line drawn through this point is the indefinite perspective of the corner of the building which meets the horizontal plane at *d*.

On the vertical lines passing through the points A and B, lay off the distance from the ground to the eave-trough. From the upper extremity of the vertical line through A, draw a line to the vanishing point K, and note the point in which it intersects the vertical line through *e'*: the part intercepted between the vertical lines through A and *e'* is the perspective of the lower line of the eave-trough that is horizontally projected in the line Ae. The line joining the upper extremities of the vertical lines through B and e', is the perspective of the lower line of the eave-trough of which Be is the projection, and the perspective would, if produced, pass through H. From the upper extremity of the vertical line through B, draw a line to K. The part cut off by the vertical line through f' is the perspective of the lower line of the eave-trough, of which Bf is the horizontal projection. Through the upper extremity of the vertical line through A, a line has been drawn to K; the part of this line intercepted between the vertical lines through f' and d', is the perspective of all that can be seen of the lower line of the eave-trough, whose horizontal projection is gd.

From A draw a line to H, and from b a line to S. The vertical line drawn through a'', their point of intersection, is the indefinite perspective of the corner of the house which meets the horizontal plane at a. From the upper extremity of the vertical line through A, draw a line to H. The part cut off by the vertical line through a'' is the perspective of the lower line of the eave-trough of which Aa is the horizontal projection.

On the vertical lines through A and B, lay off the distance to the lower line of the water-table; then the width of the water-table; then find the perspectives of its upper and lower lines, in the same manner as we have already found the perspectives of the lines of the eave-trough. On the vertical line through A, lay off from the line DE, the distance to the upper face of the lower window-sill, and draw through the point a horizontal line, and also a line to K. The horizontal line is the projection on the perspective plane of the lower line of the windows of the first story, and the line to K is its perspective.

From the point in which the perspective meets the vertical line through e', draw a line to H, and it will be the perspective of the lower line of the window, in that part of the building whose horizontal projection is B*e*: this line intersects the vertical line through B, at the same point in which it is intersected by the projection of the lower line of the windows. The line drawn from this point to K is the indefinite perspective of the lower line of the windows, corresponding to the part of the house of which B*f* is the horizontal projection. By similar constructions we can find the perspectives of the upper horizontal lines of the windows, and also the perspectives of the horizontal lines which are tangent to their semicircular arches.

In finding the perspectives of the vertical lines of the windows, it should not be forgotten that they are parallel to the perspective plane, and consequently, their perspectives will be parallel to the lines themselves. Through the lower extremity of the vertical line whose horizontal projection is *h*, draw a line perpendicular to the perspective plane. It will pierce the perspective plane in the projection of the lower line of the windows. From the point at which it pierces, draw a line to S; its intersection with the perspective of the horizontal line before referred to, is the perspective of one point of the vertical bounding line of the window; and the indefinite line can be drawn, since it is vertical. The other vertical lines are found by constructions entirely similar, and all of them are limited by the perspectives, which have already been found, of the horizontal lines of the building.

To find the perspective of the roofs of the house. Produce *nm*, the horizontal projection of the line in which the side roofs intersect, until it meets the ground line at E. Draw EE′ perpendicular to the ground line, and make it equal to the height of the intersection of the roofs above the ground. Then E' will be the point in which the intersection of the roofs pierces the perspective plane—the horizontal line through E' is the projection, on the perspective plane, of the intersection of the roofs, and the line drawn from E' to the vanishing point K is its indefinite perspective. The point of which k is the horizontal projection is vertically projected at k'. From k' draw a line to S': the point k'', in which it intersects the line $E'K$, is the perspective of the point (k, k'). The perspectives of the points (m, m'), (n, n'), and (p, p'), are found by similar constructions.

If through the point in which the vertical line through A meets the horizontal plane of the eaves of the house, a line be drawn to k', it will be the projection on the perspective plane of the line in which the side and front-end roofs, intersect.

The horizontal lines of the water-table, of the windows, and of the eave-trough, corresponding to the end of the house, have a common vanishing point H; and the perspectives of the vertical lines of the windows are found in the same manner as those in the front of the house.

To find the perspectives of the chimneys. From the horizontal line $n'E'$ lay off a distance above it equal to the height of the tops of the chimneys above the intersection of the side roofs, and through the point so determined, draw the horizontal line rs. Produce the horizontal lines of the cornices which intersect at either of the angles that are seen, until they meet the perspective plane, which they will do in the line rs. Since H and K are the vanishing points of these lines, their perspectives can be drawn, and the point in which any two of them, that pass through the same angular point of the cornice, intersect, is the perspective of an angular point of the cornice. The perspectives of the lines in which the faces of the chimneys intersect the roof of the house and the perspectives of their vertical edges are easily found.

99. Although the perspective of every point can be found rigorously, yet the smaller parts of the building can generally be made with sufficient accuracy, without making a separate construction for each of them. Thus, after we have found the bounding lines of the windows, the window-sills, the caps and casings may be made very accurately without the aid of a geometrical construction. Having found the perspectives of the doors and steps, we may make the railing without a particular construction for each vertical bar. After having found the lower line of the eave-trough, we may draw the outer upper line, always observing the rules of perspective, and the general symmetry of the picture.

In looking obliquely upon a building, the casings of the doors and windows on the side nearest the eye are not seen, while those on the other side are very distinct.

To find the perspective of the shadows cast upon the house. Let R be the projection of a ray of light on the horizontal plane, R′ its projection on the perspective plane, and R″ the position of its horizontal projection, after it has been revolved to correspond with the horizontal projection of the building. The rays of light being nearly parallel to the perspective plane, their vanishing point will be so far distant from the centre of the picture that we cannot use it conveniently in finding the shadows. We therefore adopt other methods.

In all the constructions, we must bear in mind that the house is behind the perspective plane, and therefore, when a ray of light is drawn through any point, its horizontal projection must not be drawn parallel to R, but to R″, which makes the same angle with the ground line, but is differently inclined. Let us first find the shadow cast by the upper line of the end eave-trough, on the end of the house.

From t draw a line perpendicular to the ground line, and make vt' equal to the distance from the ground to the upper line of the eave-trough. Through (t, t') draw a ray of light; it pierces the end wall of the house in the point (u, u'), which is a point in the line of shadow. From the point in which uu' meets the ground line, draw a line to S; the point in which it intersects AH is the perspective of the point u. Through this point draw a vertical line, and it will be the perspective of the vertical line passing through (u, u'). From u', draw a line to S; the point in which it intersects the vertical line before found, is the perspective of one point of the required line of shadow. But since the shadow is parallel to the line itself, its vanishing point is at H; therefore, the line drawn through H, and the point found, is the indefinite perspective of the line of shadow.

Through the highest point, which is in the light, of the corner of the house which meets the horizontal plane at A, draw a ray of light. Such ray will pierce the face Bc of the building in a point which is horizontally projected at i. The vertical line drawn through i is the shadow cast on the wall by the corner of the house which meets the horizontal plane at A. The perspective of this vertical line of shadow, which is easily found, is the perspective of the shadow cast by the corner of the house towards the source of light. From the upper extremity of this shadow, the shadow is cast for a short distance by the upper line of the end eave-trough; then by the upper line of the front eave-trough; and the shadow terminates in the line of shadow cast by the upper line of the eave-trough belong-ing to the part of the house B*e*; the latter shadow being parallel in space to the shadow cast by the end eave-trough on the end of the house.

The shadow of one of the chimneys only is seen. This shadow is found by drawing rays of light through the extremities of the lines casting it; finding the points in which they pierce the roof, and determining the perspectives of those points.

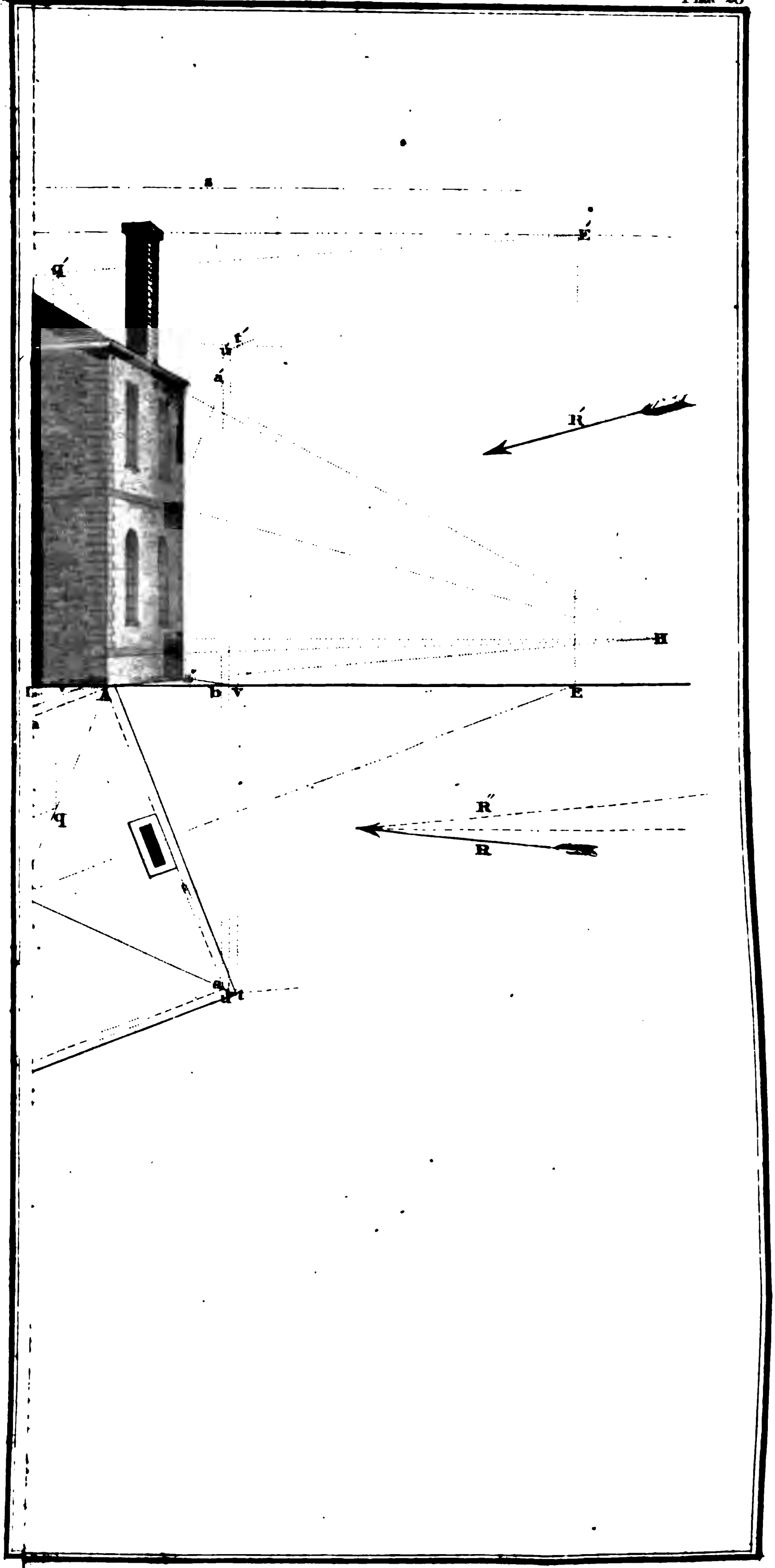
100. It is, perhaps, too obvious to require illustration, that the projections in Descriptive Geometry, the Orthographic and Stereographic projections of the sphere, are but particular methods of perspectives.

If the point of sight be at an infinite distance from the perspective plane, the perspectives of objects are the same as their orthographic projections.

101. If the plane of any circle of the sphere be assumed for the perspective plane, the point of sight being at the pole, the perspective representation of the sphere upon this plane will in nowise differ from its stereographic projection.

102. We have thus far represented objects on plane surfaces only. Their perspectives may however be made on other surfaces, and the principles which have been explained will require but a slight modification to render them applicable to the case in which the perspective is made on any surface whatever.

Suppose, for example, it were required to construct the perspectives of the different circles of the earth on the surface of a tangent cylinder, the point of sight being at the centre. If we suppose the equator to be the circle of contact, the perspectives of the meridians will be elements of the cylinder, and the perspective of any other circle is found by constructing the intersection of the cone, of which the circle is the base, and the



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point of sight the vertex, with the surface of the tangent cylinder. Having thus found the perspectives of all the circles of the sphere, if we develop the surface of the tangent cylinder on a plane, its development will be a map of the earth. Mercator's chart is constructed on these principles.

103. Panoramic views, which often exhibit entire cities, are generally constructed on the surface of a vertical cylinder, the eye of the spectator being in the axis. When the perspective is accurately made, and viewed from the right point, the deception is perfect. The houses seem to stand out from the canvass on which they are drawn; the streets have the aspect of bustle and business, and one feels himself transported into a populous city, and mingling in its affairs.

THE END.

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